MATH 521A: Abstract Algebra

Homework 11: Due Dec. 13

1. Let R be a commutative ring, and let I, J be ideals of R. Prove that $I \cap J$ is an ideal of R.

2. Find I, J, ideals of $\mathbb{Z}$, such that $I \cup J$ is not be an ideal of $\mathbb{Z}$.

3. Let R be a commutative ring, and let I, J be ideals of R. Prove that $I + J = \{a + b : a \in I, b \in J\}$ is an ideal of R.

4. Let R be a commutative ring, and let I, J be ideals of R. Prove that $IJ = \{\sum_{i=1}^{k} a_i b_i : k \in \mathbb{N}, a_i \in I, b_i \in J\}$ is an ideal of R.

5. Find I, J, ideals of $\mathbb{Z}[x]$, such that $K = \{ab : a \in I, b \in J\}$ is not an ideal of $\mathbb{Z}[x]$. Hint: Neither ideal can be principal.

6. Let R be a commutative ring, and let I, J be ideals of R. Prove that $IJ \subseteq I \cap J$.

7. Let R be a commutative ring, and suppose that $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ is an infinite tower of ideals, each contained in the next. Set $I = \cup_{j=1}^{\infty} I_j$. Prove that I is an ideal.

8. Find all ideals in $\mathbb{Z}_8$, and then use the first isomorphism theorem to find all homomorphic images of $\mathbb{Z}_8$.

9. Prove that every ideal in $\mathbb{Z}$ is principal.

10. Use the first isomorphism theorem to find all homomorphic images of $\mathbb{Z}$.