1. Prove that $T = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$ is a subfield of $\mathbb{R}$. Note that $\mathbb{Q}$ is a subfield of $T$.

We first note that $0 = 0 + 0\sqrt{2} \in T$. Second, we calculate $(a + b\sqrt{2}) - (a' + b'\sqrt{2}) = (a - a') + (b - b')\sqrt{2} \in T$, so $T$ is closed under subtraction. Lastly, we calculate $(a+b\sqrt{2})(a'+b'\sqrt{2}) = (aa' + 2bb') + (ab' + ba'\sqrt{2}) \in T$, so $T$ is closed under multiplication. Hence $T$ is a subring of $\mathbb{R}$. Lastly, if $a + b\sqrt{2}$ is nonzero, then neither is $a - b\sqrt{2}$, and so neither is their product $a^2 - 2b^2$. We calculate $(a + b\sqrt{2})\left(\frac{a}{a^2 - 2b^2} + \frac{-b}{a^2 - 2b^2}\sqrt{2}\right) = \frac{a^2 - 2b^2}{a^2 - 2b^2} \pm \frac{0}{a^2 - 2b^2}\sqrt{2} = 1$. Hence $T$ is a field.

2. Let $F, G$ be rings such that $\mathbb{Q}$ is a subring of each. Suppose $\phi : F \to G$ is a (ring) isomorphism. Prove that, for every $a \in \mathbb{Q}$, in fact $\phi(a) = a$.

First, we recall from Thm 3.10 (or prove from scratch) that $\phi(0) = 0$ and $\phi(1) = 1$. Second, for $n \in \mathbb{N}$, we have $\phi(n) = \phi(1 + 1 + \cdots + 1) = \phi(1) + \phi(1) + \cdots + \phi(1) = 1 + 1 + \cdots + 1 = n$. Third, for $m, n \in \mathbb{N}$, we have $n = \phi(n) = \phi(n\frac{m}{m} + \frac{n}{m} + \cdots + \frac{n}{m}) = \phi(n\frac{m}{m}) + \phi(n\frac{n}{m}) + \cdots + \phi(n\frac{n}{m}) = m\phi(n\frac{m}{m})$. Dividing both sides by $m$, we get $\frac{n}{m} = \phi\left(n\frac{m}{m}\right)$. Lastly, for $m, n \in \mathbb{N}$, we have $0 = \phi(0) = \phi(n\frac{m}{m} + \frac{n}{m}) = \phi\left(n\frac{m}{m}\right) + \phi\left(\frac{n}{m}\right) = n\phi\left(n\frac{m}{m}\right) + \frac{n}{m} = \frac{n}{m} + \phi\left(\frac{n}{m}\right)$, so $-\frac{n}{m} = \phi\left(\frac{n}{m}\right)$.

3. Prove that $R = \mathbb{Q}[x]/(x^2 - 2)$ is not isomorphic to $S = \mathbb{Q}[x]/(x^2 - 3)$. Hint: problem 2.

We argue by contradiction; suppose $\phi : R \to S$ were an isomorphism. Both fields have $\mathbb{Q}$ as subrings, so we may apply problem 2 to conclude that $\phi([2]_R) = [2]_S$. We now calculate $0_S = 0_R = \phi([x^2 - 2]_R) = \phi([x]_R^2 - [2]_R) = \phi([x]_R)^2 - \phi([2]_R) = \phi([x]_R)^2 - [2]_S$, and hence $\phi([x]_R)^2 = [2]_S$. Now, $\phi([x]_R) = \phi(ax + b)_S$, so $[2]_S = ([ax + b]_S)^2 = [a^2x^2 + 2abx + b^2]_S = [2abx + (3a^2 + b^2)]_S$. Hence we have some $a, b \in \mathbb{Q}$ satisfying $2ab = 0$ and $3a^2 + b^2 = 2$. The first equation means that $a = 0$ (which leads to $b = \pm\sqrt{2} \notin \mathbb{Q}$), or $b = 0$ (which leads to $a = \pm\sqrt{2/3} \notin \mathbb{Q}$). Hence we have a contradiction.

4. Prove that $R = \mathbb{Q}[x]/(x^2 - 2)$ is isomorphic to $S = \{a + b\sqrt{2} : a, b \in \mathbb{Q}\}$.

The natural isomorphism to try is $\phi : [bx + a]_R \mapsto a + b\sqrt{2}$. There are four things to check. We calculate $\phi([bx + a] + [bx' + a']) = \phi(\{(b + b')x + (a + a')\}) = (a + a') + (b + b')\sqrt{2} = (a + b\sqrt{2}) + (a' + b'\sqrt{2}) = \phi([bx + a]) + \phi([bx' + a'])$. The slightly tricky one is $\phi([bx + a][b'x + a']) = \phi([bb'x^2 + (ba' + ab')x + aa']) = \phi([ba' + ab']x + (2bb' + aa')] = (2bb' + aa') + (ba' + ab')\sqrt{2} = (a + b\sqrt{2})(a' + b'\sqrt{2}) = \phi([bx + a])\phi([bx' + a'])$. Suppose that $\phi([bx + a]) = \phi([bx' + a'])$. Then $a + b\sqrt{2} = a' + b'\sqrt{2}$, so $a = a'$, $b = b'$ and $[bx + a] = [bx' + a']$. This proves injectivity. Lastly, let $a + b\sqrt{2} \in S$. We have $\phi([bx + a]) = [bx + a]$. This proves surjectivity.

5. Set $F = \mathbb{Q}[x]/(x^3 - x + 1)$. Prove that $f(x) = x^3 - x + 1$ splits in $F$. That is, find three distinct roots of $f(x)$ in $F$.

The easiest root is $[x]$; we have $f([x]) = [x^3 - x + 1] = [0]$ in $F$. To find the others takes
a bit of trial and error. We have \( f([x+1]) = ([x+1])^3-(x+1)+1 = [x^3+3x^2+2x+1] = [x^3-x+1] = [0] \) in \( F \). Lastly, we have \( f([x-1]) = ([x-1])^3-(x-1)+1 = [x^3-3x^2+2x+1] = [x^3-x+1] = [0] \) in \( F \).

6. Prove that \( \{1, \sqrt{2}, i, i\sqrt{2}\} \) is linearly independent over \( \mathbb{Q} \).

Suppose we have \( 0 = a1 + b\sqrt{2} + ci + di\sqrt{2} \), for some \( a, b, c, d \in \mathbb{Q} \). First, we consider the real and imaginary parts separately; this tells us that \( 0 = a1 + b\sqrt{2} \) and \( 0 = ci + di\sqrt{2} \). Dividing the latter by \( i \), we get \( 0 = c1 + d\sqrt{2} \). Now, if \( b \) is nonzero, we have \( \sqrt{2} = \frac{-a}{b} \), a contradiction since \( \sqrt{2} \notin \mathbb{Q} \). Hence \( b = 0 \) and hence \( a = 0 \). Similarly, \( c = d = 0 \).

7. Set \( R = \mathbb{Q}(\sqrt{2}) \), and \( S = R(i) \). Determine \([R : \mathbb{Q}], [S : R], \) and \([S : \mathbb{Q}]\).

The minimal polynomial for \( \sqrt{2} \) over \( \mathbb{Q} \) is \( x^2 - 2 \); this is irreducible by Eisenstein (\( p = 2 \)). Hence \([R : \mathbb{Q}] = 2 \). Now, \( i \in S \) but \( i \notin R \), so \([S : R] > 2 \). A polynomial whose root is \( i \), over \( R \) (and over \( \mathbb{Q} \)) is \( x^2 + 1 \). If this were reducible, then \([S : R] < 2 \), which we know isn’t true, so this is irreducible. Hence \([S : \mathbb{Q}] = [S : R][R : \mathbb{Q}] = 2 \cdot 2 = 4 \).

8. Prove that \( x^4 - 2x^2 + 9 \) is the minimal polynomial for \( i + \sqrt{2} \) over \( \mathbb{Q} \). (remember to prove irreducibility)

First, we evaluate \((i + \sqrt{2})^4 - 2(i + \sqrt{2})^2 + 9 = 0 \), so \( i + \sqrt{2} \) is a root. Since the polynomial has real coefficients, the conjugate, \( i - \sqrt{2} \), is also a root. Since the polynomial is even, the negatives of these are also roots. Hence, over \( \mathbb{C} \), the polynomial factors as \((x - i - \sqrt{2})(x - i + \sqrt{2})(x + i - \sqrt{2})(x + i + \sqrt{2}) \). None of these four linear factors are in \( \mathbb{Q}[x] \), but it’s possible it has two quadratic factors. If so, the linear factors would break into two pairs. However, \((x - i - \sqrt{2})(x + i + \sqrt{2}) = x^2 - 2i\sqrt{2} - 1 \notin \mathbb{Q}[x] \), and \((x - i + \sqrt{2})(x + i - \sqrt{2}) = x^2 - 2ix - 3 \notin \mathbb{Q}[x] \), and \((x - i - \sqrt{2})(x + i - \sqrt{2}) = x^2 - 2\sqrt{2}x + 3 \notin \mathbb{Q}[x] \). Hence the polynomial is irreducible over \( \mathbb{Q} \). Since it is monic, it is the minimal polynomial for all four of these roots.

9. Set \( T = \mathbb{Q}(i + \sqrt{2}) \), and let \( R, S \) be as in problem 7. Prove that \( 1, \sqrt{2}, i, i\sqrt{2} \) are all in \( T \), so \( S \subseteq T \).

First, \( 1 \in T \) since \( \mathbb{Q} \in T \). Second, \((i + \sqrt{2})^2 = 1 + 2i\sqrt{2} \notin T \) (since \( 1 \in T \)) and hence \( i\sqrt{2} \in T \) (since \( 2 \in T \)). Now, \((i + \sqrt{2})(i\sqrt{2}) = 2i - \sqrt{2} \in T \). Hence \((i + \sqrt{2}) + (2i - \sqrt{2}) = 3i \in T \), and hence \( i \in T \) (since \( 3 \in T \)). Lastly, \((i + \sqrt{2}) - i = \sqrt{2} \in T \). Since each basis element of \( S \) is in \( T \), all of \( S \) is in \( T \).

10. Let \( R, S, T \) be as in problems 7 and 9. Determine \([T : \mathbb{Q}] \), and hence \([T : S] \). What can we conclude about \( S, T \)?

We have \([T : \mathbb{Q}] = 4 \), since the minimal polynomial is of degree 4. But also \([T : \mathbb{Q}] = [T : S][S : \mathbb{Q}] = [T : S]4 \). Hence \([T : S] = 1 \), and in fact \( S = T \).