MATH 521A: Abstract Algebra
Homework 8: Due Nov. 2

1. Find all irreducible polynomials of degree at most 3 in $\mathbb{Z}_2[x]$.
2. Express $x^4 - 4$ as a product of irreducibles in $\mathbb{Q}[x], \mathbb{R}[x], \mathbb{C}[x], \mathbb{Z}_3[x]$.
3. Prove that $x^3 - 2$ is irreducible in $\mathbb{Z}_7[x]$.
4. Find all roots of $x^2 + 11$ in $\mathbb{Z}_{12}[x]$.
5. Express $x^{11} - x$ as a product of irreducibles in $\mathbb{Z}_{11}[x]$. Hint: FLT.
6. Suppose $F \subseteq K$ are both fields. Let $f \in F[x] \subseteq K[x]$. Suppose that f is irreducible in $K[x]$. Prove that f is also irreducible in $F[x]$.
7. Suppose $p(x)$ is irreducible in $F[x]$, and $a \in F$ is nonzero. Prove that $ap(x)$ is also irreducible.
8. Let $f(x) = a_0 + a_1x + \cdots + a_{n-1}x^{n-1} + a_nx^n \in F[x]$. Define $\overline{f}(x) = a_n + a_{n-1}x + \cdots + a_1x^{n-1} + a_0x^n \in F[x]$. Suppose that $c \neq 0$ is a zero of $f(x)$. Prove that c^{-1} is a zero of $\overline{f}(x)$.
9. Let $a \in F$ and define $\phi_a : F[x] \to F$ via $\phi_a : f(x) \mapsto f(a)$. Prove that ϕ_a is a surjective (ring) homomorphism.
10. Define $\mathbb{Q}[\sqrt{2}] = \{r_0 + r_1\sqrt{2} + r_2(\sqrt{2})^2 + \cdots + r_n(\sqrt{2})^n : n \geq 0, r_i \in \mathbb{Q}\}$. Note that this definition differs from our previous one for $\mathbb{Q}[\sqrt{2}]$ (although they can be proved equivalent). Consider the function $\phi : \mathbb{Q}[x] \to \mathbb{Q}[\sqrt{2}]$ via $\phi : f(x) \mapsto f(\sqrt{2})$. Prove that ϕ is a (ring) homomorphism, is surjective, and is not injective.