MATH 521A: Abstract Algebra
Homework 6: Due Oct. 12

1. Let R, S be rings. Consider the *embedding* map $f : R \to R \times S$ given by $f : r \mapsto (r, 0_S)$. Prove that f is a homomorphism.

2. Let R, S be rings. Consider the *projection* map $f : R \times S \to R$ given by $f : (r, s) \mapsto r$. Prove that f is a homomorphism.

3. We call a ring element x *idempotent* if $x^2 = x$. Let R, S be rings, and $f : R \to S$ a homomorphism. Suppose $x \in R$ is idempotent. Prove that $f(x)$ is idempotent.

4. We call a ring element x *nilpotent* if there is some $n \in \mathbb{N}$ such that $x^n = 0$. Let R, S be rings, and $f : R \to S$ a homomorphism. Suppose $x \in R$ is nilpotent. Prove that $f(x)$ is nilpotent.

5. Let R, S be rings, and $f : R \to S$ a homomorphism. Define the kernel of f, $Kerf = \{r \in R : f(r) = 0_S\}$. Prove that $Kerf$ is a subring of R.

6. Let R, S be rings, and $f : R \to S$ a homomorphism. Prove that f is injective (one-to-one) if and only if $Kerf = \{0_R\}$.

7. Let R, S be rings, and $f : R \to S$ a homomorphism. Suppose that S_1 is a subring of S. Prove that $f^{-1}(S_1) = \{r \in R : f(r) \in S_1\}$ is a subring of R.

8. Let R, S, T be rings, and $f : R \to S$, $g : S \to T$ two homomorphisms. Prove that $g \circ f : R \to T$ is a homomorphism.

9. Let R, S be rings, and $f : R \to S$ an isomorphism. Let $g = f^{-1}$, i.e. for all $r \in R$, $g(f(r)) = r$ and for all $s \in S$, $f(g(s)) = s$. Prove that $g : S \to R$ is an isomorphism.

10. Let $S = \{\left(\begin{smallmatrix} a & 2b \\ b & a \end{smallmatrix}\right) : a, b \in \mathbb{Z}\}$, which is a subring of $M_{2,2}(\mathbb{Z})$ (two-by-two matrices with integer entries). Prove that S is isomorphic to $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$, a subring of $\mathbb{R}$.

11. Recall the ring from HW4 #6: R has ground set $\mathbb{Z}$ and operations $\oplus, \odot$ defined as:

$$a \oplus b = a + b - 1, \quad a \odot b = a + b - ab$$

Prove that R is isomorphic to $\mathbb{Z}$.

12. Recall the ring from HW4 #7: R has ground set $\mathbb{Z}$ and operations $\oplus, \odot$ defined as:

$$a \oplus b = a + b - 1, \quad a \odot b = ab - a - b + 2$$

Prove that R is isomorphic to $\mathbb{Z}$.