MATH 521A: Abstract Algebra
Homework 5: Due Oct. 5

1. Let R be a ring, with $a, b \in R$. Prove that if ab is a left zero divisor[1], then either a or b must be a left zero divisor.

2. Let R be a ring, with nonzero $a \in R$. Prove that if a is not a left zero divisor, then a may be cancelled on the left. That is, if $ab = ac$, then $b = c$.

3. Let R be a ring with identity, with $a \in R$. Suppose that a is a unit. Prove that multiplicative inverses are two-sided, i.e. $ab = 1$ if and only if $ba = 1$.

4. Let R be a ring with identity, with $a \in R$. Suppose that a is a unit. Prove that multiplicative inverses are unique, i.e. if $ab = 1$ and $ac = 1$, then $b = c$.

5. Let R and $S \subseteq R$ both be rings with identity. Find an example where $1_S \neq 1_R$.

6. Let R and $S \subseteq R$ both be integral domains. Prove that $1_S = 1_R$.

7. Consider $R = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) : a \in \mathbb{Z},\ b, c \in \mathbb{Q}\}$, a subring of the 2×2 matrix ring over $\mathbb{R}$. Determine the units and left zero divisors of R.

8. Let R be a ring, and let $S_1, S_2, \ldots$ be infinitely many subrings of R. Prove that their mutual intersection $T = \bigcap_{i \geq 1} S_i$ is a subring of R.

9. Let R_1, R_2 be rings. Suppose that S_1 is a subring of R_1, and S_2 is a subring of R_2. Prove that $S_1 \times S_2$ is a subring of $R_1 \times R_2$.

10. Let R be a ring with the property that for all $x \in R$, $x^2 = x$. Prove that each element of R is its own negative, and that R is commutative.

[1] For a general ring R, we say nonzero $a \in R$ is a left zero divisor if there is some nonzero $x \in R$ with $ax = 0$. We say a is a right zero divisor if $xa = 0$. We say a is a zero divisor if either of these two holds.