1. For ring $R$, define its center $Z(R) = \{a \in R : \forall r \in R, ra = ar\}$. Prove that $Z(R)$ is a subring of $R$.

2. For ring $R$, and $x \in R$, define the centralizer of $x$ $C_x(R) = \{a \in R : ax = xa\}$. Prove that $C_x(R)$ is a subring of $R$.

3. Let $R$ be a ring, and $S_1, S_2$ both subrings of $R$. Prove or disprove that $S_1 \cup S_2$ must be a subring of $R$.

4. Let $R = \mathbb{Z}, U = 17\mathbb{Z}$, two rings. Suppose $U \subseteq V \subseteq R$, and $V$ is a ring. Prove that $V = U$ or $V = R$.

5. Prove that $\mathbb{Q}[\sqrt{2}] = \{a + b\sqrt{2} + c\sqrt{4} : a, b, c \in \mathbb{Q}\}$ is a commutative ring with identity.

6. For ring $R$ with $a, b \in R$, we say $a$ is a left divisor of $b$ if there is some $c \in R$ with $ac = b$. Suppose that $R$ is a field, $a, b \in R$, and $a \neq 0$. Prove that $a$ is a left divisor of $b$.

7. With left divisors defined as in problem 6, let $R = M_{2,2}(\mathbb{R}), a = (\begin{smallmatrix} 1 & 3 \\ 0 & 2 \end{smallmatrix}), b = (\begin{smallmatrix} 2 & 3 \\ 0 & 2 \end{smallmatrix})$. Determine whether or not $a$ is a left divisor of $b$, and whether or not $b$ is a left divisor of $a$.

8. For the next four problems, let $X = \{1, 2, 3, \ldots, 100\}$, and let the power set of $X$, denoted $\mathcal{P}(X)$, be the set of all subsets of $X$. Let $R$ have ground set $\mathcal{P}(X)$, with operations $a \odot b = a \cap b$ and $a \oplus b = a \Delta b = (a \setminus b) \cup (b \setminus a) = (a \cup b) \setminus (a \cap b)$.

   Prove that $R$ is a commutative ring with identity.

9. For $R$ as in problem 8, for all $a \in R$, define $\overline{a} = 1_R \oplus a$. Prove that (i) $a \odot a = a$; (ii) $a \oplus a = 0$; (iii) $\overline{a} = X \setminus a$ (the complement of $a$); (iv) $a \odot \overline{a} = 1_R$; (v) $a \odot \overline{a} = 0_R$.

10. For $R$ as in problem 8, define $f : R \to \mathbb{Z}_2$ via $f : x \mapsto \begin{cases} [0] & 7 \notin x \\ [1] & 7 \in x \end{cases}$. Prove that $f$ is a homomorphism.

11. For $X, \mathcal{P}(X)$ as in problem 8, define $S$ with ground set $\mathcal{P}(X)$ and operations $a \odot b = a \cap b$ and $a \oplus b = a \cup b$. Prove that $S$ is not a ring.

12. Let $R$ be a ring such that $x^2 = 0$ for all $x \in R$. For all $a, b \in R$, prove that $a$ commutes with $ab + ba$.

13. Suppose $R$ has all the ring axioms except $a + b = b + a$. Prove that axiom from the others.

14. Let $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$. Consider the function $f : \mathbb{Z}_{mn} \to \mathbb{Z}_m \times \mathbb{Z}_n$ given by $f : [a]_{mn} \mapsto ([a]_m, [a]_n)$, proved a homomorphism in class. Prove that $f$ is a bijection.

15. For ring $R, x \in R$, and $n \in \mathbb{N}$, we say $x$ has additive order $n$ if $x + x + \cdots + x = 0_R$, and for $m < n$ we have $x + x + \cdots + x \neq 0_R$. We write this $\text{ord}_R(x) = n$. Suppose we have a homomorphism $f : R \to S$, and $x \in R$ has an additive order. Prove that $\text{ord}_S(f(x))|\text{ord}_R(x)$, i.e. the order of $f(x)$ divides the order of $x$.

16. With additive order as defined in problem 15, suppose that $x \in R$ has an order and $f : R \to S$ is an isomorphism. Prove that $\text{ord}_R(x) = \text{ord}_S(f(x))$.

17. Let $R = M_{2,2}(\mathbb{Z})$ and $S = \mathbb{Z}$. Prove or disprove that trace $R \to S$ given by trace $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) = a + d$ is a homomorphism.

18. Let $R = M_{2,2}(\mathbb{Z})$ and $S = \mathbb{Z}$. Prove or disprove that det $R \to S$ given by det $(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) = ad - bc$ is a homomorphism.

19. Let $R$ be the set of all continuous real-valued functions defined on $[0, 1]$, with the natural ring operations $(f + g)(x) = f(x) + g(x), (fg)(x) = f(x)g(x)$. Prove that $R$ is a commutative ring with $1_R$.

20. Let $R$ be the ring from problem 19, and define $\phi : R \to \mathbb{R}$ as $\phi : f \mapsto f(1/2)$. Prove that $\phi$ is a homomorphism, and find its kernel and image.