Math 254 Spring 2014 Exam 6 Solutions

1. Carefully state the definition of "dependent". Give two dependent sets, drawn from $M_{2,3}$.

   A set of vectors is dependent if there is a nondegenerate linear combination yielding the zero vector. Three examples: $\{\left(\begin{smallmatrix}0&0&0\\0&0&0\end{smallmatrix}\right)\}, \{\left(\begin{smallmatrix}1&0&0\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}2&0&0\\0&0&0\end{smallmatrix}\right)\}, \{\left(\begin{smallmatrix}1&1&1\\0&0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&1&1\\1&1&1\end{smallmatrix}\right), \left(\begin{smallmatrix}0&0&0\\1&1&1\end{smallmatrix}\right)\}$

2. Set $V = \mathbb{R}^3$. Consider all subspaces of V. What are the possible dimensions of these subspaces? Give an example subspace, for each possible dimension.

   Possible dimensions are $0, 1, 2, 3$. The only 0-dimensional subspace is $\{(0,0,0)\}$, the zero vector by itself. A 1-dimensional subspace is $Span(\{(1,3,7)\})$. A two-dimensional subpace is $Span(\{(1,3,7),(1,2,1)\})$. The only 3-dimensional subpace is V itself.

3. Set $V = \mathbb{R}^2$, and consider the basis $B = \{(1,2),(4,7)\}$. Find the change-of-basis matrix Q_{BE}, and use this to find $[(2,1)]_B$.

   We first write down $Q_{EB} = \left(\begin{smallmatrix}1&4\\2&7\end{smallmatrix}\right)$, which has B as its columns. We then have $Q_{BE} = Q_{EB}^{-1} = \left(\begin{smallmatrix}-7&4\\2&-1\end{smallmatrix}\right)$, which we can find using the 2×2 formula or the general algorithm. Lastly, we have $[(2,1)]_B = Q_{BE}[(2,1)]_E = \left(\begin{smallmatrix}-7&4\\2&-1\end{smallmatrix}\right)\left(\begin{smallmatrix}2\\1\end{smallmatrix}\right) = \left(\begin{smallmatrix}-10\\3\end{smallmatrix}\right)$

The remaining problems are both in the polynomial vector space $V = Span(B)$, whose basis is $B = \{x^2, y^2, z^2, xy, xz, yz\}$. Consider $U = Span((x+y)^2, (x-z)^2, (y-z)^2)$, a subspace of V.

4. Find the dimension of U.

   Note that $[(x+y)^2]_B = (1,1,0,2,0,0), [(x-z)^2]_B = (1,0,1,0,-2,0), [(y-z)^2]_B = (0,1,1,0,0,-2)$. We row reduce $\left(\begin{smallmatrix}1&1&0&2&0&0\\1&0&1&0&-2&0\\0&1&1&0&0&-2\end{smallmatrix}\right) \to \left(\begin{smallmatrix}1&1&0&2&0&0\\0&-1&1&-2&-2&0\\0&1&1&0&0&-2\end{smallmatrix}\right) \to \left(\begin{smallmatrix}1&1&0&2&0&0\\0&-1&1&-2&-2&0\\0&0&2&-2&-2&-2\end{smallmatrix}\right)$ via $R_2 - R_1 \to R_2$ and $R_3 + R_2 \to R_3$. The row echelon form matrix has three pivots, so the dimension of U is 3.

5. Determine whether or not $(x+y+z)^2$ is in U.

   Note that $[(x+y+z)^2]_B = (1,1,1,2,2,2)$. We row reduce $\left(\begin{smallmatrix}1&1&0&2&0&0\\0&-1&1&-2&-2&0\\0&0&2&-2&-2&-2\\1&1&1&2&2&2\end{smallmatrix}\right) \to \left(\begin{smallmatrix}1&1&0&2&0&0\\0&-1&1&-2&-2&0\\0&0&2&-2&-2&-2\\0&0&1&0&2&2\end{smallmatrix}\right) \to \left(\begin{smallmatrix}1&1&0&2&0&0\\0&-1&1&-2&-2&0\\0&0&2&-2&-2&-2\\0&0&0&1&3&3\end{smallmatrix}\right)$ via $R_4 - R_1 \to R_4$, $R_4 - \frac{1}{2}R_3 \to R_4$. The resulting row echelon matrix has four pivots, so $(x+y+z)^2$ is *not* in U. If you like you may also begin with $\left(\begin{smallmatrix}1&1&0&2&0&0\\1&0&1&0&-2&0\\0&1&1&0&0&-2\\1&1&1&2&2&2\end{smallmatrix}\right)$, which gets to the same place but takes a bit longer.