1. Carefully state the definition of “dependent”. Give two dependent sets, drawn from $M_{2,3}$.

A set of vectors is dependent if there is a nondegenerate linear combination yielding the zero vector. Three examples: $\{(0,0,0), (1,0,0), (1,1,0), (1,1,1)\}$

2. Set $V = \mathbb{R}^3$. Consider all subspaces of $V$. What are the possible dimensions of these subspaces? Give an example subspace, for each possible dimension.

Possible dimensions are 0, 1, 2, 3. The only 0-dimensional subspace is $\{(0,0,0)\}$, the zero vector by itself. A 1-dimensional subspace is $\text{Span}\{(1,3,7)\}$. A two-dimensional subspace is $\text{Span}\{(1,3,7),(1,2,1)\}$. The only 3-dimensional subspace is $V$ itself.

3. Set $V = \mathbb{R}^2$, and consider the basis $B = \{(1,2),(4,7)\}$. Find the change-of-basis matrix $Q_{BE}$, and use this to find $[(2,1)]_B$.

We first write down $Q_{EB} = \begin{pmatrix} 1/2 & 1/7 \\ 1/2 & -1/7 \end{pmatrix}$, which has $B$ as its columns. We then have $Q_{BE} = Q_{EB}^{-1} = \begin{pmatrix} -7/2 & 4/7 \\ 2 & -1 \end{pmatrix}$, which we can find using the $2 \times 2$ formula or the general algorithm. Lastly, we have $[(2,1)]_B = Q_{BE}[(2,1)]_E = \left( \begin{pmatrix} -7/2 & 4/7 \\ 2 & -1 \end{pmatrix} \right) (\begin{pmatrix} 2 \\ 1 \end{pmatrix}) = (\begin{pmatrix} -10 \\ 3 \end{pmatrix})$

The remaining problems are both in the polynomial vector space $V = \text{Span}(B)$, whose basis is $B = \{x^2, y^2, z^2, xy, xz, yz\}$. Consider $U = \text{Span}(x+y)^2, (x-z)^2, (y-z)^2)$, a subspace of $V$.

4. Find the dimension of $U$.

Note that $[(x+y)^2]_B = (1,1,0,2,0,0)$, $[(x-z)^2]_B = (1,0,1,0,-2,0)$, $[(y-z)^2]_B = (0,1,1,0,0,-2)$. We row reduce $\begin{pmatrix} 1 & 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 0 \end{pmatrix}$ via $R_2 \rightarrow R_2 - R_1$ and $R_3 + R_2 \rightarrow R_3$. The row echelon form matrix has three pivots, so the dimension of $U$ is 3.

5. Determine whether or not $(x+y+z)^2$ is in $U$.

Note that $[(x+y+z)^2]_B = (1,1,1,2,2,2)$. We row reduce $\begin{pmatrix} 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & -1 & -2 & -2 & -2 & 2 \\ 0 & 0 & 1 & 0 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 0 & 1 & 3 & 3 \end{pmatrix}$ via $R_4 - R_1 \rightarrow R_4, R_4 - \frac{1}{2}R_3 \rightarrow R_4$. The resulting row echelon matrix has four pivots, so $(x+y+z)^2$ is not in $U$. If you like you may also begin with $\begin{pmatrix} 1 & 1 & 0 & 2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & -2 & 0 & 0 \end{pmatrix}$, which gets to the same place but takes a bit longer.