Math 254 Spring 2014 Exam 2a Solutions

1. Carefully state the definition of the “polynomial space” \( P(t) \). Give two example vectors from \( P_1(t) \).

\( P(t) \) is the vector space consisting of all polynomials in the variable \( t \). Two vectors from \( P_1(t) \) are \( 3, 3 + 2t \).

2. List, in any order, the three elementary operations that leave unchanged the solution set to a system of linear equations.

E1: Interchange two equations. E2: Multiply an equation by a nonzero constant. E3: Add a multiple of one equation to another.

3. Solve the following system of equations using Gaussian elimination and back-substitution.

\[
\begin{align*}
2y - z &= 1 \\
x - y + z &= 1 \\
2x + y + 2z &= 2
\end{align*}
\]

Step 1: \( E_1 \leftrightarrow E_2 \). Step 2: \( -2E_1 + E_3 \rightarrow E_3 \). Step 3: \( -1.5E_2 + E_3 \rightarrow E_3 \). We now back-substitute: \( z = -1 \), then \( 2y + 1 = 1 \) so \( y = 0 \). Lastly \( x - 0 - 1 = 1 \) so \( x = 2 \).

4. Consider the system of equations \( \{2x - 2y = 4, 4x + ay = b\} \). For which values of \( a, b \) does this have exactly one solution (and what is it)? For which values of \( a, b \) does this have no solution? For which values of \( a, b \) does this have infinitely many solutions?

We solve: \( -2E_1 + E_2 \rightarrow E_2 \) gives us \( \{2x - 2y = 4, (4 + a)y = (-8 + b)\} \), back-substitute \( y = \frac{b-8}{a+4} \) and \( 2x - 2 \left( \frac{b-8}{a+4} \right) = 4 \) so \( x = 2 + \frac{b-8}{a+4} = \frac{2a+b}{a+4} \). This is a unique solution, provided \( a \neq -4 \). If \( a = -4 \), then the second equation is \( 0 = b - 8 \). If \( a = -4 \) and \( b = 8 \), there are infinitely many solutions; if \( a = -4 \) and \( b \neq 8 \), then there are no solutions.

5. Find a set of points in the plane that have infinitely many lines of best fit. Be sure to justify your answer.

One solution is any single point, e.g. \( \{(2, 3)\} \). This gives system \( \{b + 2m = 3, 2b + 4m = 6\} \), which has infinitely many solutions since the second equation is twice the first. Another solution is the empty set, which gives system \( \{0b + 0m = 0, 0b + 0m = 0\} \). Others are possible, e.g. \( \{(0, 3), (0, 5)\} \), with system \( \{2b + 0m = 8, 0b + 0m = 0\} \).