1. Carefully state the definition of $M_{m,n}$. Give a set of two vectors from $M_{3,2}$.

The vector space $M_{m,n}$ consists of all matrices with $m$ rows and $n$ columns (with real entries). A set of two vectors from $M_{3,2}$ is \{\[
\begin{pmatrix}
1 & 1 \\
2 & 2 \\
3 & 3 \\
\end{pmatrix}, \begin{pmatrix}
0 & 0 \\
1 & 0 \\
0 & 0 \\
\end{pmatrix}
\}\}.

2. For $n \geq 1$, the “checkerboard” matrix $A$ has entries $a_{i,j} = (-1)^{i+j}$. Find $|A|$.

For an example, we consider $n = 3$. Here we have $A = \begin{pmatrix}
1 & -1 & 1 \\
-1 & 1 & -1 \\
-1 & 1 & -1 \\
\end{pmatrix}$; we see that adding the second row to the first gives an all-zero row, so $|A| = 0$. In fact this is always true, since $(-1)^{1+j} + (-1)^{2+j} = (-1)^{1+j} - (-1)^{1+j} = 0$ for all $j$. Hence $|A| = 0$ for all checkerboard matrices, provided that $n \geq 2$. The case $n = 1$ is special: $A = (1)$, so $|A| = 1$.

Alternate solution: For $n \geq 3$, the first and third rows of $A$ are the same, so $|A| = 0$.

We consider the special cases $n = 1, 2$. For $n = 1$, $|A| = 1$; for $n = 2$, $|A| = 0$.

The remaining three problems all concern matrix $M = \begin{pmatrix}
0 & 1 & -2 \\
1 & 2 & 3 \\
-3 & 5 & 2 \\
\end{pmatrix}$.

3. Find $|M|$ using the formula for $3 \times 3$ determinants.

We use the mnemonic that begins by repeating the first two columns of $M$. This gives $\begin{pmatrix}
0 & 1 & -2 \\
1 & 2 & 3 \\
-3 & 5 & 2 \\
\end{pmatrix}$. We now calculate $|M| = 0 \cdot 2 \cdot 5 + 1 \cdot 2 \cdot 3 + (-2) \cdot 1 \cdot (-3) - (2 \cdot 2 \cdot (-2)) - ((-3) \cdot 3 \cdot 0) - (5 \cdot 1 \cdot 1) = 0 + 6 + 6 + 8 + 0 - 5 = 15$.

4. Find $|M|$ with a Laplace expansion on the first column.

We have $|M| = 0 \begin{vmatrix}
\frac{2}{3} & 3 & -1 \\
1 & -3 & 5 \\
2 & 1 & -2 \\
\end{vmatrix}$ + $2 \begin{vmatrix}
1 & -2 \\
\frac{1}{3} & 3 \\
2 & 1 \\
\end{vmatrix} = 0 - 1(1.5 - (-3)(-2)) + 2(1.3 - 2(-2)) = 1 + 14 = 15$.

5. Find $|M|$ using elementary row operations to make it block triangular.

Method 1: Swap rows 1,3 to get $\begin{pmatrix}
2 & -3 & 5 \\
1 & 2 & 3 \\
0 & 1 & -2 \\
\end{pmatrix}$. This multiplies the determinant by $-1$.

Then $R_2 - \frac{1}{2} R_1 \rightarrow R_2$ gives $\begin{pmatrix}
2 & -3 & 5 \\
0 & 3.5 & 0.5 \\
0 & 1 & -2 \\
\end{pmatrix}$. This does not change the determinant. We now have a block triangular matrix, whose determinant is $|2| \begin{vmatrix}
3.5 & 0.5 \\
1 & -2 \\
\end{vmatrix} | = 2 \cdot (7 - 0.5) = -15$. Hence $|A| = -((-15)) = 15$.

Method 2: We continue one more step, via $R_3 = R_3 - \frac{2}{7} R_2$ to get $\begin{pmatrix}
2 & -3 & 5 \\
0 & 3.5 & 0.5 \\
0 & 0 & -4 \frac{5}{7} \\
\end{pmatrix}$. This is now fully triangular (and also block triangular). We now find its determinant by multiplying along the diagonal, i.e. $2 \cdot \frac{7}{2} \cdot -\frac{15}{7} = -15$. Hence $|A| = -((-15)) = 15$. 
