Math 254 Spring 2014 Exam 1 Solutions

1. Carefully state the definition of “linear function space”. Give two example vector spaces.

A linear function space is the (linear) span of some set of variables. Examples include Span(x), Span(x, y, z), Span(s, t).

2. State whether or not each of the following are defined; if defined state what type of object is the result. Note: “3-vector” means “vector from \( \mathbb{R}^3 \”).

(a) (column 3-vector)(row 3-vector)

Juxtaposition means matrix multiplication. The result is a 3 \( \times \) 3 matrix.

(b) (column 3-vector) \cdot (row 3-vector)

\cdot means dot product (of vectors). The result is a scalar.

(c) (column 3-vector) \times (row 3-vector)

\times means cross product (of 3-vectors only). The result is a 3-vector.

(d) (3-vector)+(3-vector)

Two vectors from the same vector space may always be added. The result is a 3-vector.

(e) (3 \times 1 matrix)+(1 \times 3 matrix)

Matrices of different dimensions may not be added; this sum is not defined.

3. Let \( u = (1, 1, 1, 3) \), \( v = (-1, 0, x, 1) \). Determine all possible \( x \), if any, that make vectors \( u,v \) \( \in \mathbb{R}^4 \) orthogonal.

We calculate \( u \cdot v = -1 + 0 + x + 3 = 2 + x \). The vectors \( u,v \) are orthogonal exactly when \( u \cdot v = 0 \), that is when \( 2 + x = 0 \), or \( x = -2 \).

4. Let \( u = (1, 2, 3) \). Find any vector \( v \) such that \( u \times v = (0, 0, 0) \).

We have \( \|u \times v\| = \|u\|\|v\|\sin \theta \), which we want to be \( \|(0, 0, 0)\| = 0 \). Hence we want \( \theta = 0 \), or \( v \) to be parallel to \( u \). Thus any multiple of \( u \) will work: \( v \) can be \((1, 2, 3)\) or \((2, 4, 6)\) or \((-1, -2, -3)\) or \((0, 0, 0)\).

5. Calculate the matrix product \( \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} \). Hint: it’s nice.

We calculate \( \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix} \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & -1/2 \end{bmatrix} = \begin{bmatrix} 1/4+3/4 & -\sqrt{3}/4+\sqrt{3}/4 \\ \sqrt{3}/4-\sqrt{3}/4 & 1/4+1/4 \end{bmatrix} = [1 0 1]. \) Later we will learn that this isn’t a surprise; we have demonstrated that rotating counterclockwise by an angle of \( -\frac{2\pi}{3} \), followed by rotating counterclockwise by an angle of \( \frac{2\pi}{3} \), is the same as not doing anything at all.