1. Carefully state the definition of “polynomial space” \( P(t) \). Give two different bases for \( P_1(t) \).

The polynomial space \( P(t) \) consists of all polynomials, with real coefficients, in the variable \( t \). Two bases for \( P_1(t) \) are \( \{1, t\} \) and \( \{1 + t, 1 - t\} \).

2. Let \( V \) denote the set of all symmetric \( 2 \times 2 \) matrices. Set \( E = \{(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix})\} \).

Prove that \( E \) is a basis for \( V \).

We first prove that \( E \) is independent: If \( a e_1 + b e_2 + c e_3 = 0 \), then \( (\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}) = (\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix}) \), so \( a = b = c = 0 \). Hence no nondegenerate linear combination yields \( 0 \).

Solution 1: Let \( (\begin{smallmatrix} a \\ b \\ c \end{smallmatrix}) \in V \). We take \( a e_1 + b e_2 + c e_3 \), and see that it equals the desired matrix. Hence \( E \) is spanning, and hence \( E \) is a basis.

Solution 2: \( V \neq M_{2,2} \) so \( \text{dim}(V) \leq 3 \). But \( E \) is independent and \( |E| = 3 \), so \( E \) is maximal spanning, and is thus a basis.

The remaining three problems concern the vector space \( V = \{(\begin{smallmatrix} a \\ b \\ d \end{smallmatrix}) : a, b, d \in \mathbb{R}\} \) and its basis \( E = \{(\begin{smallmatrix} 1 \\ 0 \\ 0 \end{smallmatrix}), (\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}), (\begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix})\} \).

3. Set \( B = \{(\begin{smallmatrix} 0 & -2 \\ -2 & 1 \end{smallmatrix}), (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}), (\begin{smallmatrix} 0 & -1 \\ -1 & 1 \end{smallmatrix})\} = \{b_1, b_2, b_3\} \). Compute \([b_1]_E; [b_2]_E; [b_3]_E\), and use these to prove that \( B \) is a basis for \( V \).

We have \([b_1]_E = (0, -2, 1), [b_2]_E = (0, 1, 0), [b_3]_E = (1, 0, -1)\). Putting these as rows, we get \( \begin{pmatrix} 0 & -2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \). After \( R_1 + 2R_2 \rightarrow R_1 \), \( R_1 \leftrightarrow R_3 \), we get \( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \). This is in row echelon form, and has 3 pivots, so the row space of the original matrix is 3-dimensional. Hence \( \text{dim}(\text{Span}(B)) = 3 = \text{dim}(V) \), and thus \( \text{Span}(B) = V \), and \( B \) is a basis for \( V \).

4. Set \( C = \{(\begin{smallmatrix} 1 \\ 3 \\ 2 \end{smallmatrix}), (\begin{smallmatrix} 2 \\ 6 \\ 4 \end{smallmatrix}), (\begin{smallmatrix} 1 \\ 1 \\ 1 \end{smallmatrix}), (\begin{smallmatrix} 3 \\ 4 \end{smallmatrix})\} = \{c_1, c_2, c_3, c_4\} \). Compute \([c_1]_E; [c_2]_E; [c_3]_E; [c_4]_E\), and use these to find a basis for \( \text{Span}(C) \).

We have \([c_1]_E = (1, 3, 2), [c_2]_E = (2, 6, 4), [c_3]_E = (1, 1, 1), [c_4]_E = (5, 3, 4)\). Putting these as rows, we get \( \begin{pmatrix} 1 & 3 & 2 \\ 2 & 6 & 4 \\ 1 & 1 & 1 \end{pmatrix} \), which has row echelon form \( \begin{pmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \). Hence \( \text{Span}(C) \) has basis \( \{(\begin{smallmatrix} 1 \\ 3 \\ 2 \end{smallmatrix}), (\begin{smallmatrix} 0 \\ 2 \\ 1 \end{smallmatrix})\} \).

5. For \( B \) as in (3), calculate \( Q_{BE} \), and use this to compute \( [\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}]_B \).

We now put the \([b_1]_E; [b_2]_E; [b_3]_E\) as columns, to get \( Q_{EB} \). We calculate \( Q_{BE} = Q_{EB}^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \). Since \( [\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}]_E = (1, 2, 3)^T \), we calculate \( [\begin{smallmatrix} 1 \\ 2 \\ 3 \end{smallmatrix}]_B = Q_{BE}(1, 2, 3)^T = (4, 10, 1)^T \).

That is, \( \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 4b_1 + 10b_2 + 1b_3 \), which is easily double-checked if desired.