Math 254 Fall 2014 Exam 1 Solutions

1. Carefully state the definition of "polynomial space in s". Give a set of two vectors, drawn from $P_1(s)$.

The polynomial space in s, denoted $P(s)$, is the set of all polynomials with real coefficients with variable s. A set of two vectors drawn from $P_1(s)$ is $\{2+3s, 4-2s\}$.

2. Let $u = \begin{bmatrix} 2 & 17 & -1 \end{bmatrix}$. For each of the following, specify what type of object v must be for the expression to be defined. If the expression can never be defined, write NONE.

(a) $u^T v + uv^T$
NONE: v must be a matrix, since juxtaposition denotes matrix multiplication. For $u^T v$ to make sense, v must have one row. For uv^T to make sense, v must have three columns. But now $u^T v$ is a 3×3 matrix, while uv^T is a 1×1 matrix, which cannot be added.

(b) $u \cdot v$
v must be a 3-vector, i.e. a vector from $\mathbb{R}^3$. Row or column doesn't matter.

(c) $u \times v$
v must be a 3-vector, i.e. a vector from $\mathbb{R}^3$. Row or column doesn't matter.

(d) $(u \cdot v) \times u$
NONE: $u \cdot v$ is a scalar, but a cross product demands a 3-vector.

(e) $u \cdot (v \times u)$
v must be a 3-vector, i.e. a vector from $\mathbb{R}^3$. Row or column doesn't matter.

3. Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Compute ABA^T.

$ABA^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 4 \\ 3 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 5 & 18 \end{pmatrix}$.

4. Let $u = (1, 0, -1), v = (0, 2, 3)$. Calculate $u \times v$.
METHOD 1: $u \times v = (\mathbf{i} - \mathbf{k}) \times (2\mathbf{j} + 3\mathbf{k}) = 2(\mathbf{i} \times \mathbf{j}) + 3(\mathbf{i} \times \mathbf{k}) - 2(\mathbf{k} \times \mathbf{j}) - 3(\mathbf{k} \times k) = 2(\mathbf{k}) + 3(-\mathbf{j}) - 2(-\mathbf{i}) = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} = (2, -3, 2)$.
METHOD 2: $\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 2 & 3 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 2\mathbf{k} - 0\mathbf{k} - (-2)\mathbf{i} - 3\mathbf{j} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k} = (2, -3, 2)$.

5. Let $u = (1, 2, 3), v = (x, y, 0)$. Find values for x, y such that u, v are orthogonal *and also* simultaneously $\|v\| = 1$.
The orthogonality condition means that $0 = u \cdot v = x + 2y$. The other condition means that $\sqrt{x^2 + y^2 + 0^2} = 1$, i.e. $x^2 + y^2 = 1$. We plug $x = -2y$ into the latter to get $(-2y)^2 + y^2 = 1$, or $4y^2 + y^2 = 1$ or $5y^2 = 1$ or $y^2 = \frac{1}{5}$. Hence $y = \pm\frac{1}{\sqrt{5}}$, and $x = -2y$. This gives two possible answers: $(\frac{-2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0)$ and its negative.