# Math 254 Fall 2014 Exam 0 Solutions

1. Carefully state the definition of a linear function space. Give two examples, which must be vector spaces

   A linear function space on a set of variables is the set of all linear functions of those variables. Some examples are $Span(x, y), Span(r, s, t), Span(x)$.

2. Carefully state the definition of "nondegenerate span". Give a set from $\mathbb{R}^2$ whose nondegenerate span is *different* from its span.

   The nondegenerate span of a set of vectors is the set of all linear combinations of those vectors, *except* the all-zero linear combination. Examples: $\{(1, 2)\}$, $\{(1, 2), (1, 3)\}$.

3. Consider the subset of $\mathbb{R}^2$ given by $S = \{(a, b) : a \geq b\}$. Prove that $S$ not a subspace of $\mathbb{R}^2$.

   We need closure under vector addition and scalar multiplication. The latter fails; we need a specific counterexample, such as $(2, 1) \in S$ but $(-4)(2, 1) = (-8, -4) \notin S$. As it happens, $S$ is closed under vector addition, but that is irrelevant here.

4. Consider the matrix space $M_{2,2}$. Prove that $\left\{\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 1 & 5 \end{smallmatrix}\right), \left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right)\right\}$ is independent.

   Suppose the set is dependent, then $a\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) + b\left(\begin{smallmatrix} 1 & 0 \\ 1 & 5 \end{smallmatrix}\right) + c\left(\begin{smallmatrix} 2 & 0 \\ 0 & 3 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$, for some $a, b, c$ not all zero. We add the LHS to get $\left(\begin{smallmatrix} a+b+2c & a \\ b & a+5b+3c \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)$. This gives us the system of equations $\{a + b + 2c = 0, a = 0, b = 0, a + 5b + 3c = 0\}$ which has only solution $a = b = c = 0$. This contradiction proves independence.

5. Consider the polynomial space $P_2(t)$. Prove that $\{t^2 - 1, t - 1\}$ is not spanning.

   We need to find one vector and prove it is not in the span. Many choices will work, for example $t^2$. Suppose $t^2 = a(t^2 - 1) + b(t - 1) = at^2 + bt + (-a - b)$. This gives us the system of equations $\{a = 1, b = 0, -a - b = 0\}$. There are no solutions to this system, hence this set is not spanning.