Math 254 Fall 2014 Exam 0 Solutions

1. Carefully state the definition of a linear function space. Give two examples, which must be vector spaces.

A linear function space on a set of variables is the set of all linear functions of those variables. Some examples are \( \text{Span}(x, y), \text{Span}(r, s, t), \text{Span}(x) \).

2. Carefully state the definition of “nondegenerate span”. Give a set from \( \mathbb{R}^2 \) whose nondegenerate span is different from its span.

The nondegenerate span of a set of vectors is the set of all linear combinations of those vectors, except the all-zero linear combination. Examples: \{ (1, 2) \}, \{ (1, 2), (1, 3) \}.

3. Consider the subset of \( \mathbb{R}^2 \) given by \( S = \{ (a, b) : a \geq b \} \). Prove that \( S \) not a subspace of \( \mathbb{R}^2 \).

We need closure under vector addition and scalar multiplication. The latter fails; we need a specific counterexample, such as \((2, 1) \in S \) but \((-4)(2, 1) = (-8, -4) \notin S \). As it happens, \( S \) is closed under vector addition, but that is irrelevant here.

4. Consider the matrix space \( M_{2,2} \). Prove that \{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 5 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \} \) is independent.

Suppose the set is dependent, then \( a \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} + b \begin{pmatrix} 1 & 0 \\ 1 & 5 \end{pmatrix} + c \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \), for some \( a, b, c \) not all zero. We add the LHS to get \( \begin{pmatrix} a+b+2c \\ b \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \). This gives us the system of equations \( \begin{cases} a + b + 2c = 0 \\ b = 0 \end{cases} \) which has only solution \( a = b = c = 0 \). This contradiction proves independence.

5. Consider the polynomial space \( P_2(t) \). Prove that \{ \begin{pmatrix} t^2 - 1, t - 1 \end{pmatrix} \} \) is not spanning.

We need to find one vector and prove it is not in the span. Many choices will work, for example \( t^2 \). Suppose \( t^2 = a(t^2 - 1) + b(t - 1) = at^2 + bt + (-a - b) \). This gives us the system of equations \( \{ a = 1, b = 0, -a - b = 0 \} \). There are no solutions to this system, hence this set is not spanning.