Math 254 Fall 2013 Exam 8 Solutions

1. Carefully state the definition of “vector space”. Give two three-dimensional examples.

A vector space is a collection of objects (called vectors), a set of scalars (typically \( \mathbb{R} \)), and a way to add vectors and multiply vectors by scalars. Two familiar three-dimensional examples are \( \mathbb{R}^3 \) and \( P_2(t) \).

2. Carefully state the definition of “linear transformation”. Give two examples on \( P_2(t) \).

A linear transformation is a function \( f \) from a vector space \( U \) to a vector space \( V \), satisfying:

(1) For all \( u, v \in U \), \( f(u+v) = f(u) + f(v) \), and (2) For all \( u \in U \) and \( k \in \mathbb{R} \), \( f(ku) = kf(u) \).

Many examples are possible such as \( f(p(t)) = p(t) \) (identity), \( f(p(t)) = -p(t) \), \( f(at^2 + bt + c) = bt^2 + (a + c)t + a \).

3. Consider the mapping \( f : P_1(t) \rightarrow \mathbb{R}^3 \) given by \( f(a + bt) = (a, a + b, 2b) \). Determine, with justification, whether or not \( f \) is linear.

1. Let \( a + bt, a' + b't \) be two arbitrary vectors in \( P_1(t) \). We have \( f(a + bt) + f(a' + b't) = (a, a+b, 2b) + (a', a'+b', 2b') = (a+a', a+b+a'+b', 2b+2b') = (a+a', a+b+a'+b', 2b+b') = f((a+a') + (b+b')(t)) = f(a + bt + (a' + b't)). \) This is the first required property.

2. Let \( a + bt \) be arbitrary in \( P_1(t) \), and let \( k \in \mathbb{R} \). We have \( kf(a + bt) = k(a, a + b, 2b) = (ka, ka + kb, 2kb) = f(ka + kbt) \). This is the second required property, so the answer is YES.

4. Consider the linear mapping \( g : \mathbb{R}^3 \rightarrow P_2(t) \) given by \( g((a, b, c)) = a + (b + c)t + at^2 \). Find a basis for the kernel of \( g \), and find a basis for the image of \( g \).

If \( (a, b, c) \) is in the kernel of \( g \), then \( g((a, b, c)) = a + (b + c)t + at^2 = 0 \), so \( a = 0, b + c = 0, a = 0 \). This is a one-dimensional space, with basis \( \{(0, 1, -1)\} \).

By the rank-nullity theorem, \( \text{dim(Im g)} + \text{dim(Ker g)} = \text{dim(\mathbb{R}^3)} \), so \( \text{dim(Im g)} = 2 \) and any basis for \( \text{Im g} \) will consist of two (linearly independent) vectors. One example is \( \{1 + t^2, t\} \).

5. Let \( f, g \) be as in problems 3.4. Consider the linear mapping \( h : P_1(t) \rightarrow P_2(t) \) given by \( h = g \circ f \). Calculate \( h(1+2t) \), and determine (with justification) whether \( h \) is an isomorphism.

We have \( h(1+2t) = (g \circ f)(1+2t) = g(f(1+2t)) = g(1, 3, 4) = 1 + 7t + t^2 \). The linear map \( h \) is NOT an isomorphism, and here are two possible explanations why not:

1. We calculated in Problem 4 that \( \text{dim(Im g)} = 2 \), so \( \text{dim(Im h)} \leq 2 \). But \( \text{dim(P}_2(t)) = 3 \), so \( h \) cannot be onto.

2. By the rank-nullity theorem \( \text{dim(P}_1(t)) = \text{dim(Ker h)} + \text{dim(Im h)} \). Because \( \text{dim(P}_1(t)) = 2 \) and \( \text{dim(Ker h)} \geq 0 \), we must have \( \text{dim(Im h)} \leq 2 < \text{dim(P}_2(t)) \), so \( h \) cannot be onto.

Extra: Consider the linear mapping \( f : M_{2,2} \rightarrow M_{2,2} \) given by \( f(A) = \frac{1}{2}(A + A^T) \). Find a basis for the kernel of \( f \), and find a basis for the image of \( f \). Are either of these spaces familiar?

Suppose \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \) is in the kernel of \( f \). Then \( 0 = f(\begin{pmatrix} a & b \\ c & d \end{pmatrix}) = \begin{pmatrix} a & (b+c)/2 \\ (b+c)/2 & d \end{pmatrix} \), so \( a = 0 = d \) and \( \frac{1}{2}(b + c) = 0 \). This is a one-dimensional subspace with basis \( \{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\} \), also known as the set of all skew-symmetric 2 \( \times \) 2 matrices. By the rank-nullity theorem, \( \text{dim(Im f)} = 3 \); by applying \( f \) to each of \( \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \), we find a basis for \( \text{Im f} \) of \( \{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\} \). This subspace is also known as the set of all symmetric 2 \( \times \) 2 matrices.