Math 254 Fall 2013 Exam 7 Solutions

1. Carefully state the definition of "basis". Give two examples from $P_1(t)$.

   A basis is a set of vectors that is independent and spanning. Two examples are $\{1, t\}$ and $\{2, 1+t\}$.

2. Suppose that V is a vector space with some inner product $\langle \cdot, \cdot \rangle$. Recall the derived norm is given by $\|u\| = \sqrt{\langle u, u \rangle}$. Prove that $\|kv\| = |k|\|v\|$ for all $v \in V$ and for all $k \in \mathbb{R}$.

   Let $v \in V$ and $k \in \mathbb{R}$ be arbitrary. We calculate $\|kv\| = \sqrt{\langle kv, kv \rangle} = \sqrt{k\langle v, kv \rangle} = \sqrt{k^2\langle v, v \rangle} = |k|\sqrt{\langle v, v \rangle} = |k|\|v\|$. In the second and third inequalities we used the linearity of an inner product in the first and second coordinate, respectively.

   The remaining problems all concern the inner product on $\mathbb{R}^3$ defined by $\langle x, y \rangle_A = x^T A y$, where A is the positive definite matrix $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Set $u = (0, 1, 1)^T$, $v = (1, 1, 0)^T$.

3. Find the projection of u along v and the angle between u, v.

   We first calculate $\langle u, u \rangle_A = 3$, $\langle u, v \rangle_A = 2$, $\langle v, v \rangle_A = 2$. Now $Proj_v u = \frac{\langle u,v \rangle_A}{\langle v,v \rangle_A} v = \frac{2}{2} v = v$. The angle between u, v is given by $\cos\theta = \frac{\langle u,v \rangle_A}{\|u\|\|v\|} = \frac{2}{\sqrt{3}\sqrt{2}} = \sqrt{\frac{2}{3}}$, so $\theta = \cos^{-1}(\sqrt{\frac{2}{3}})$.
   (This turns out to not be a nice angle, so we can't simplify. It's ≈ 0.615 radians or $\approx 35.3°$.)

4. Find an orthonormal basis for $Span(u, v)$.

   We use Gram-Schmidt; let $w_1 = v$, $w_2 = u - Proj_v u$. We can use what we found in (3) to get $w_2 = u - v = (-1, 0, 1)$. $\{w_1, w_2\}$ is an orthogonal basis; to make it orthonormal we must divide each by its length. $\|w_1\| = \sqrt{2}$, as found in (3). We calculate $\|w_2\| = 1$. Hence $\{(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0), (-1, 0, 1)\}$ is an orthonormal basis for $Span(u, v)$.

5. Find a basis for $Span(u, v)^\perp$.

   Solution 1: Since $Span(u, v)$ is 2-dimensional and $\mathbb{R}^3$ is 3-dimensional, we need a single vector orthogonal to both u, v. We may start with any vector not in $Span(u, v)$, say $r = (1, 0, 0)$. We first calculate $r' = r - Proj_{w_1} r = r - \frac{1}{2} w_1 = (1/2, -1/2, 0)$. We now calculate $r'' = r' - Proj_{w_2} r' = r' - 0$. Any multiple of this works as well, so we may as well clear the fractions, and take basis $\{(1, -1, 0)\}$.

   Solution 2: We again seek a single vector $r = (a, b, c)$. Since $\langle r, u \rangle_A = 0$, we conclude that $a + b + 2c = 0$. Since $\langle r, v \rangle_A = 0$, we conclude that $a + b + c = 0$. Hence $c = 0$ and $a = -b$. We can pick a arbitrarily as 1; this gives basis $\{(1, -1, 0)\}$.

Extra: We continue our adventures with the inner product on $\mathbb{R}^3$ defined by $\langle x, y \rangle_A = x^T A y$, where A is the positive definite matrix $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$. Your task is to find an orthonormal basis for $\mathbb{R}^3$.

Let's start with the standard basis $\{e_1, e_2, e_3\}$ and apply Gram-Schmidt. $f_1 = e_1$, $f_2 = e_2 - \frac{\langle e_2, f_1 \rangle_A}{\langle f_1, f_1 \rangle_A} f_1$. What joy, $\langle e_2, f_1 \rangle_A = 0$ so $f_2 = e_2$! Now $f_3 = e_3 - \frac{\langle e_3, f_1 \rangle_A}{\langle f_1, f_1 \rangle_A} f_1 - \frac{\langle e_3, f_2 \rangle_A}{\langle f_2, f_2 \rangle_A} f_2 = e_3 - 1f_1 - 0f_2 = (-1, 0, 1)$. We now rescale f_1, f_2, f_3; as it happens $\|f_1\| = \|f_2\| = \|f_3\| = 1$ so in fact $\{(1, 0, 0), (0, 1, 0), (-1, 0, 1)\}$ is an orthonormal basis.