Math 254 Fall 2013 Exam 3 Solutions

1. Carefully state the definition of "spanning". Give two examples from $P_2(t)$.

   A set of vectors is spanning if their span is the entire vector space. Some possible examples: $\{1, t, t^2\}, \{1, 1+t, 1+t^2\}, \{1, 2t, 3t^2\}, \{1, t, 2t, 3t^2\}$.

2. Suppose that B is an invertible 5×5 matrix. Prove that B^2 is also invertible.

   Solution 1: Because B is invertible there is some B^{-1} with $BB^{-1} = I$. We have $B^2(B^{-1})^2 = B(BB^{-1})B^{-1} = BIB^{-1} = BB^{-1} = I$, so the product of B^2 with matrix $(B^{-1})^2$ gives I.

   Solution 2: By Thm 3.7, because B is invertible there are elementary matrices $E_1, E_2, \ldots, E_k$ such that $B = E_1E_2\cdots E_k$. Then $B^2 = (E_1E_2\cdots E_k)^2$, a product of elementary matrices. Hence by Thm 3.7 again, B is invertible.

   The remaining three problems all concern the matrix $A = \begin{bmatrix} 0&0&1\\0&1&0\\1&0&0 \end{bmatrix}$.

3. Be sure to justify your answers to the following questions. (a) Is A diagonal? (b) Is A triangular? (c) Calculate $tr(A)$. (d) Calculate A^T. (e) Is A orthogonal?

   (a) No, because $a_{1,3}$ (or $a_{3,1}$) is nonzero. (b) No, because both of $a_{1,3}$ and $a_{3,1}$ are nonzero. (c) $tr(A) = 0 + 1 + 0 = 1$ (d) We reflect along the main diagonal to get $A^T = \begin{bmatrix} 0&0&1\\0&1&0\\1&0&0 \end{bmatrix}$. This happens to be A again. (e) Yes, because $AA^T = I$.

4. Find a symmetric matrix B and a skew-symmetric matrix C such that $A = B + C$.

   We have $B = \frac{1}{2}(A + A^T)$ and $C = \frac{1}{2}(A - A^T)$. As it happens, $B = A = \begin{bmatrix} 0&0&1\\0&1&0\\1&0&0 \end{bmatrix}$ while $C = 0 = \begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$.

5. Let $D = A^2 + 2A$. Calculate D, and calculate D^{-1}.

   $D = \begin{bmatrix} 0&0&1\\0&1&0\\1&0&0 \end{bmatrix}^2 + 2\begin{bmatrix} 0&0&1\\0&1&0\\1&0&0 \end{bmatrix} = \begin{bmatrix} 1&0&0\\0&1&0\\0&0&1 \end{bmatrix} + \begin{bmatrix} 0&0&2\\0&2&0\\2&0&0 \end{bmatrix} = \begin{bmatrix} 1&0&2\\0&3&0\\2&0&1 \end{bmatrix}$. We start with $[D|I] = \begin{bmatrix} 1&0&2&1&0&0\\0&3&0&0&1&0\\2&0&1&0&0&1 \end{bmatrix}$. We first do $R_3 - 2R_1 \to R_3$ to get $\begin{bmatrix} 1&0&2&1&0&0\\0&3&0&0&1&0\\0&0&-3&-2&0&1 \end{bmatrix}$. We then do $R_1 + \frac{2}{3}R_3 \to R_1$ to get $\begin{bmatrix} 1&0&0&-1/3&0&2/3\\0&3&0&0&1&0\\0&0&-3&-2&0&1 \end{bmatrix}$. Lastly, we divide R_2 by 3 and R_3 by -3 to get $\begin{bmatrix} 1&0&0&-1/3&0&2/3\\0&1&0&0&1/3&0\\0&0&1&2/3&0&-1/3 \end{bmatrix}$. Hence $D^{-1} = \begin{bmatrix} -1/3&0&2/3\\0&1/3&0\\2/3&0&-1/3 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -1&0&2\\0&1&0\\2&0&-1 \end{bmatrix}$.

Extra: Find the inverse and LU factorization of the matrix $\begin{pmatrix} -1&2&0\\0&1&2\\1&-3&0 \end{pmatrix}$.

We start with $\begin{pmatrix} -1&2&0&1&0&0\\0&1&2&0&1&0\\1&-3&0&0&0&1 \end{pmatrix}$. We do $R_3 + R_1 \to R_3$ to get $\begin{pmatrix} -1&2&0&1&0&0\\0&1&2&0&1&0\\0&-1&0&1&0&1 \end{pmatrix}$. We then do $R_3 + R_2 \to R_3$ and $R_1 - 2R_2 \to R_1$ to get $\begin{pmatrix} -1&0&-4&1&-2&0\\0&1&2&0&1&0\\0&0&2&1&1&1 \end{pmatrix}$. We now do $R_1 + 2R_3 \to R_1$ and $R_2 - R_3 \to R_2$ to get $\begin{pmatrix} -1&0&0&3&0&2\\0&1&0&-1&0&-1\\0&0&2&1&1&1 \end{pmatrix}$. Lastly, we divide R_1 by -1 and R_3 by 2 to get $\begin{pmatrix} 1&0&0&-3&0&-2\\0&1&0&-1&0&-1\\0&0&1&1/2&1/2&1/2 \end{pmatrix}$. Hence the inverse is $\begin{pmatrix} -3&0&-2\\-1&0&-1\\1/2&1/2&1/2 \end{pmatrix}$. For the LU factorization we need only do the two steps $R_3 + R_1 \to R_3$ and $R_3 + R_2 \to R_3$. Two multipliers are 1 and one is zero (see p.119 for details). Hence $L = \begin{pmatrix} 1&0&0\\0&1&0\\-1&-1&1 \end{pmatrix}$ while $U = \begin{pmatrix} -1&2&0\\0&1&2\\0&0&2 \end{pmatrix}$.