Math 254 Fall 2013 Exam 2a Solutions

1. Carefully state the definition of "linear function space". Give an example, in two variables, and three vectors from there.

   A linear function space is the span of a set of variables. One example is $Span(x, y)$ or $\{ax+by : a, b \in \mathbb{R}\}$, which contains vectors $0, x, 2x + 3y$.

2. Carefully state the definition of "solution to a linear system of equations". Give an example of a linear system of equations that does *not* have a solution, and justify your answer.

   A solution to a linear system is an assignment of real numbers to each variable such that all the (linear) equations in the system are simultaneously true. One example of a system with no solution is $\{x + y = 3, x + y = 4\}$; we may rearrange to get $3 = 4$ which never holds for any assignment of values to x, y.

3. Solve the following system using back-substitution. Be sure to justify your calculations.

$$\begin{aligned} 6x_1 + 2x_2 - 5x_3 + x_4 &= 5 \\ 7x_2 + 2x_3 - 3x_4 &= 8 \\ 2x_3 - x_4 &= 2 \\ 2x_4 &= 8 \end{aligned}$$

   From the fourth equation, we get $x_4 = 4$. Substituting into the third equation we get $2x_3 - 4 = 2$, or $x_3 = 3$. Substituting into the second equation we get $7x_2 + 6 - 12 = 8$, or $x_2 = 2$. Lastly, substituting into the first equation we get $6x_1 + 4 - 15 + 4 = 5$, or $x_1 = 2$. Combining, our solution is $(x_1, x_2, x_3, x_4) = (2, 2, 3, 4)$.

4. Find the line of best fit for the points $\{(0, -1), (-1, 0), (1, 4), (2, 5)\}$.

   We first compute the relevant statistics. $N = 4$, $\sum x_i = 0 + (-1) + 1 + 2 = 2$, $\sum x_i^2 = 0^2 + (-1)^2 + 1^2 + 2^2 = 6$, $\sum y_i = (-1) + 0 + 4 + 5 = 8$, $\sum x_i y_i = 0 + 0 + 4 + 10 = 14$. Hence we need to solve the linear system $\{4b + 2m = 8, 2b + 6m = 14\}$. Subtracting the first equation from twice the second gives $0b + 10m = 20$, or $m = 2$. We now back-substitute to find $4b + 4 = 8$ or $b = 1$. Hence the line of best fit is $y = 2x + 1$.

5. Consider the linear equation $ax + by + cz = d$. Suppose that (x_1, y_1, z_1) and (x_2, y_2, z_2) are both solutions. Prove that $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$ is also a solution.

   We have two solutions, so $ax_1 + by_1 + cz_1 = d$ and $ax_2 + by_2 + cz_2 = d$. If we add them we get $a(x_1 + x_2) + b(y_1 + y_2) + c(z_1 + z_2) = 2d$. Dividing by 2 gives $a(\frac{x_1+x_2}{2}) + b(\frac{y_1+y_2}{2}) + c(\frac{z_1+z_2}{2}) = d$, which proves that $(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2})$ is a solution.

Extra: Find three linear equations in the variables $\{x, y, z\}$, such that each pair of these equations have an infinite number of solutions, but the three together have no mutual solutions. Be sure to justify your answer.

One method is to have all the planes be vertical, i.e. perpendicular to the x-y plane. For example, $\{x + y + 0z = 1, x - y + 0z = 1, x + 2y + 0z = 2\}$. The first two have (infinite) solution set $\{(1, 0, z) : z \in \mathbb{R}\}$, the first and last have solution set $\{(0, 1, z) : z \in \mathbb{R}\}$, and the last two have solution set $\{(4/3, 1/3, z) : z \in \mathbb{R}\}$. These three solution sets have no intersection.

Another method is to choose three arbitrary planes, such that their sum leads to a contradiction (and hence no mutual solutions). For example $\{x + y - 2z = 1, x - 2y + z = 1, -2x + y + z = 1\}$; their sum is $0 = 3$ so there are no mutual solutions. It can be shown (messy) that each pair has infinitely many solutions; intuitively no pair is parallel, so must intersect in a line.