Math 254 Fall 2013 Exam 10 Solutions

1. Carefully state the definition of "dependent". Give two examples from $P_3(t)$.

   A set of vectors is dependent if there is a nondegenerate linear combination yielding the zero vector. Some examples are $\{0\}, \{1, t, 1+t\}, \{t^2, 2t^2\}$.

2. Let $M \in M_{3,3}$. Suppose that M is similar to I_3, the identity matrix. Prove that $M = I_3$.

   Since M is similar to I_3, there is some invertible P such that $M = P^{-1} I_3 P$. We simplify, using $I_3 P = P$ and $P^{-1}P = I_3$, to get $M = P^{-1}P = I_3$.

3. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear mapping defined as $f((a,b)) = (2a, a-b)$. Let $S = \{(1,-2), (1,-1)\}$ be a basis for $\mathbb{R}^2$. Find $[f]_S$.

   We first find $P_{ES} = \begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$, by writing S as columns. We next find $P_{SE} = P_{ES}^{-1} = \begin{pmatrix} -1 & -1 \\ 2 & 1 \end{pmatrix}$. We next find $[f]_E = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$, by writing $f(e_1), f(e_2)$ as columns. Lastly, we compute $[f]_S = P_{SE}[f]_E P_{ES} = \begin{pmatrix} -5 & -4 \\ 7 & 6 \end{pmatrix}$.

4. Let V be the vector space of functions that have as a basis $S = \{e^{3t}\sin 2t, e^{3t}\cos 2t\}$. Find the matrix representation $[\frac{d}{dt}]_S$.

   Let $s_1 = e^{3t}\sin 2t, s_2 = e^{3t}\cos 2t$ for convenience. We calculate $\frac{d}{dt}s_1 = 3s_1 + 2s_2$, using the product and chain rules. We similarly calculate $\frac{d}{dt}s_2 = -2s_1 + 3s_2$. Writing these as columns, we get $[\frac{d}{dt}]_S = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$.

5. Let V be the vector space of functions that have as a basis $S = \{e^{3t}\sin 2t, e^{3t}\cos 2t\}$. Let I_v denote the identity linear mapping on V. Find the rank of $\frac{d}{dt}$, and of $(\frac{d}{dt} - I_v)$.

   From Problem 4, we have $[\frac{d}{dt}]_S = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$. One way: this matrix has determinant 13, which is nonzero, hence the matrix is nonsingular, hence $\frac{d}{dt}$ is onto, hence it has rank 2. Another way: we put $\begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix}$ into echelon form, and see two pivots, so the rank is 2.

   $[\frac{d}{dt} - I_v]_S = [\frac{d}{dt}]_S - [I_v]_S = \begin{pmatrix} 3 & -2 \\ 2 & 3 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 2 & 2 \end{pmatrix}$. We put this into echelon form, again see two pivots, so the rank is 2.

Extra: Recall that any matrix $M \in M_{3,3}$ may be written uniquely as $M = M_S + M_{SS}$, where M_S is symmetric and M_{SS} is skew-symmetric. Consider the linear map $f : M_{3,3} \to M_{3,3}$ defined as $f(M) = M_S$. Write down the standard basis E, and calculate $[f]_E$.

There are different ways to order the standard basis, which leads to different solutions to this problem. The usual way is

$$E = \left\{ \begin{pmatrix} 1&0&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\0&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\1&0&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&1&0\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&1\\0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\1&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&0\\0&0&0\\0&0&1 \end{pmatrix} \right\}.$$

Then

$$[f]_E = \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1/2&0&1/2&0&0&0&0&0\\ 0&0&1/2&0&0&0&1/2&0&0\\ 0&1/2&0&1/2&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&0&1/2&0&1/2&0\\ 0&0&1/2&0&0&0&1/2&0&0\\ 0&0&0&0&0&1/2&0&1/2&0\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix}.$$