1. Carefully state the definition of “dependent”. Give two examples from \( P_3(t) \).

A set of vectors is dependent if there is a nondegenerate linear combination yielding the zero vector. Some examples are \( \{0\}, \{1, t, 1 + t\}, \{t^2, 2t^2\} \).

2. Let \( M \in M_{3,3} \). Suppose that \( M \) is similar to \( I_3 \), the identity matrix. Prove that \( M = I_3 \).

Since \( M \) is similar to \( I_3 \), there is some invertible \( P \) such that \( M = P^{-1}I_3P \). We simplify, using \( I_3P = P \) and \( P^{-1}P = I_3 \), to get \( M = P^{-1}P = I_3 \).

3. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be the linear mapping defined as \( f((a, b)) = (2a, a - b) \). Let \( S = \{(1, -2), (1, -1)\} \) be a basis for \( \mathbb{R}^2 \). Find \([f]_S\).

We first find \( P_{ES} = \left( \begin{array}{cc} 1/2 & 1 \\ -1/2 & 1 \end{array} \right) \), by writing \( S \) as columns. We next find \( P_{SE} = P_{ES}^{-1} = \left( \begin{array}{cc} -1 & 1 \\ 2 & -1 \end{array} \right) \).

We next find \([f]_E = \left( \begin{array}{c} 2 \\ 0 \end{array} \right) \), by writing \( f(e_1), f(e_2) \) as columns. Lastly, we compute \([f]_S = P_{SE}[f]_EP_{ES} = \left( \begin{array}{cc} -5 & -4 \\ 7 & 6 \end{array} \right) \).

4. Let \( V \) be the vector space of functions that have as a basis \( S = \{e^{3t} \sin 2t, e^{3t} \cos 2t\} \). Find the matrix representation \([d/dt]_S\).

Let \( s_1 = e^{3t} \sin 2t, s_2 = e^{3t} \cos 2t \) for convenience. We calculate \( d/dt s_1 = 3s_1 + 2s_2 \), using the product and chain rules. We similarly calculate \( d/dt s_2 = -2s_1 + 3s_2 \). Writing these as columns, we get \([d/dt]_S = \left( \begin{array}{c} 3 \\ -2 \\ 2 \end{array} \right) \).

5. Let \( V \) be the vector space of functions that have as a basis \( S = \{e^{3t} \sin 2t, e^{3t} \cos 2t\} \). Let \( I_v \) denote the identity linear mapping on \( V \). Find the rank of \( d/dt \), and of \((d/dt - I_v)\).

From Problem 4, we have \([d/dt]_S = \left( \begin{array}{c} 3 \\ -2 \\ 2 \end{array} \right) \). One way: this matrix has determinant 13, which is nonzero, hence the matrix is nonsingular, hence \( d/dt \) is onto, hence it has rank 2. Another way: we put \( \begin{pmatrix} 3 \\ -2 \\ 2 \end{pmatrix} \) into echelon form, and see two pivots, so the rank is 2.

\([d/dt - I_v]_S = [d/dt]_S - [I_v]_S = \left( \begin{array}{c} 3 \\ -2 \\ 2 \end{array} \right) - \left( \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right) = \left( \begin{array}{c} 3 \\ -2 \\ 2 \end{array} \right) \). We put this into echelon form, again see two pivots, so the rank is 2.

Extra: Recall that any matrix \( M \in M_{3,3} \) may be written uniquely as \( M = M_S + M_{SS} \), where \( M_S \) is symmetric and \( M_{SS} \) is skew-symmetric. Consider the linear map \( f : M_{3,3} \to M_{3,3} \) defined as \( f(M) = M_S \). Write down the standard basis \( E \), and calculate \([f]_E\).

There are different ways to order the standard basis, which leads to different solutions to this problem. The usual way is \( E = \{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \} \). The matrix \([f]_E = \left( \begin{array}{c} \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 \end{array} \end{array} \right) \).