Math 254 Fall 2012 Exam 8 Solutions

1. Carefully state the definition of "linear transformation". Give two examples, each on $\mathbb{R}^3$.

   A linear transformation is a function f whose domain and codomain are vector spaces, that satisfies $f(u+v) = f(u) + f(v), f(ku) = kf(u)$ for all vectors u, v in its domain, and all scalars k. Many examples are possible, such as $f((a,b,c)) = (a,b,c), f((a,b,c)) = (a,0,0), f((a,b,c)) = (b,c,a)$.

2. Consider the linear map $F : M_{2,2} \to M_{2,2}$ given by $F(A) = A - A^T$. Find a basis for its kernel, and a basis for its image.

   We see that $F(\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]) = F(\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$, while $F(\left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]) = -F(\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]) = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$. Hence its image is one-dimensional, with basis $\{\left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]\}$. By the dimension theorem, its kernel is therefore three-dimensional. We have two basis elements already, and it's not too hard to find a third, giving a basis of $\{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right]\}$.

3. What are the possible ranks and nullities of linear transformations $F : \mathbb{R}^2 \to \mathbb{R}^3$? Give an example of each possible combination, and indicate which are one-to-one and which are onto.

   The dimension theorem gives just three possibilities, none of which are onto.
   rank=2, nullity=0: $F((a,b)) = (a,b,0)$. Since nullity is 0, this is one-to-one.
   rank=1, nullity=1: $F((a,b)) = (a,0,0)$. Nullity>0, so not one-to-one.
   rank=0, nullity=2: $F((a,b)) = (0,0,0)$. Also not one-to-one.

4. Consider the linear transformation $F : \mathbb{R}^2 \to P_2(x)$ given by $F((a,b)) = \int_0^x (at+b)dt$. Find its rank and nullity, and a basis for its image.

   We have $F((1,0)) = \int_0^x t dt = \frac{x^2}{2}$, and $F((0,1)) = \int_0^x 1 dt = x$. Since $\{x, \frac{x^2}{2}\}$ are independent, they form a basis of the image of F and thus the rank of F is 2. By the dimension theorem, the nullity of F is 0. Or, this could be proved directly as $\int_0^x (at+b)dt = \frac{ax^2}{2} + bx = 0$, hence $a = b = 0$.

5. Suppose that $F : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear transformation, and $F \circ F \circ F \circ F = I_2$ (identity). Prove that the nullity of F is 0, and find such an F.

   Suppose the nullity of F were greater than zero. Then there would be some nonzero vector v with $F(v) = 0$. But then $F \circ F \circ F \circ F(v) = F \circ F \circ F(0) = F \circ F(0) = F(0) = 0 \neq v = I_2(v)$. This is a contradiction, hence the nullity of F must be 0. Many examples are possible, such as $F((a,b)) = (-b,a)$ (rotation by $\frac{\pi}{4}$), $F((a,b)) = (-a,-b), F((a,b)) = (a,b)$.