1. Carefully state the definition of “vector space”. You need not write out all the properties in detail. Give two examples, each six dimensional.

A vector space is a set of vectors $V$, a field of scalars $K$, and operations of vector addition and scalar multiplication, which must be closed and satisfy eight axioms. Six-dimensional examples we’ve seen include $\mathbb{R}^6$, $P_5(t)$, $M_{2,3}(\mathbb{R})$.

2. Let $u = (2, -1, 0)$, a vector in $\mathbb{R}^3$. Compute $\|u\|_1, \|u\|_2, \|u\|_3, \|u\|_\infty$.

\[
\begin{align*}
\|u\|_1 &= |2| + | -1 | + |0| = 3. \\
\|u\|_2 &= \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}.
\end{align*}
\]

3. Consider the vector space $M_{2,2}$ with the usual inner product $\langle X, Y \rangle = tr(X^TY)$. Set $A = [\frac{1}{2} \ 0]$, and $B = [1 \ 0]$. Find $B_1, B_2$ such that $B = B_1 + B_2$, $B_1$ is a multiple of $A$, and $B_2$ is orthogonal to $A$.

We set $B_1 = proj(B, A) = \frac{\langle A, B \rangle}{\langle A, A \rangle} A = \frac{4}{10} A = \begin{bmatrix} 0.4 & 0.8 \\ 0.8 & 0.4 \end{bmatrix}$. We now find $B_2 = B - B_1 = \begin{bmatrix} 0.6 & -0.8 \\ 0.2 & 0.6 \end{bmatrix}$. If you wanted, you could double-check that $\langle A, B_2 \rangle = 0$.

The remaining two questions concern vector space $V = P_1(x)[0, 1]$, the set of polynomials of degree at most 1 on interval $[0, 1]$ with inner product $\langle u(x), v(x) \rangle = \int_0^1 u(x)v(x)dx$.

Let $S = \{s_1, s_2\}$ for $s_1(x) = \sqrt{3}x$, $s_2(x) = -3x + 2$.

4. Prove that $S$ is an orthonormal set (hence a basis).

We need to show three things: $\langle s_1, s_1 \rangle = 1$, $\langle s_2, s_2 \rangle = 1$, $\langle s_1, s_2 \rangle = 0$. We have $\langle s_1, s_1 \rangle = \int_0^1 3x^2dx = x^3|_0^1 = 1$. We have $\langle s_2, s_2 \rangle = \int_0^1 (-3x + 2)^2dx = \int_0^1 (9x^2 - 12x + 4)dx = 3x^3 - 6x^2 + 4x|_0^1 = 1$. Lastly, we have $\langle s_1, s_2 \rangle = \int_0^1 (-3x + 2)(\sqrt{3}x)dx = \sqrt{3}\int_0^1 (-3x^2 + 2x)dx = \sqrt{3}(x^3 - x^2)|_0^1 = 0$.

5. Find the Fourier coefficients of $u(x) = x + 1$ with respect to $S$.

Recall that the Fourier coefficients are the coefficients in the decomposition $u = \langle u, s_1 \rangle s_1 + \langle u, s_2 \rangle s_2$. We have $\langle u, s_1 \rangle = \int_0^1 (x + 1)\sqrt{3}x dx = \sqrt{3}\int_0^1 (x^2 + x)dx = \sqrt{3}(x^3 + \frac{x^2}{2})|_0^1 = \frac{5\sqrt{3}}{6}$. We also have $\langle u, s_2 \rangle = \int_0^1 (x + 1)(-3x + 2)dx = \int_0^1 (-3x^2 - x + 2)dx = -x^3 - \frac{x^2}{2} + 2x|_0^1 = \frac{1}{2}$.