Math 254 Fall 2012 Exam 7 Solutions

1. Carefully state the definition of "vector space". You need not write out all the properties in detail. Give two examples, each six dimensional.

   A vector space is a set of vectors V, a field of scalars K, and operations of vector addition and scalar multiplication, which must be closed and satisfy eight axioms. Six-dimensional examples we've seen include $\mathbb{R}^6, P_5(t), M_{2,3}(\mathbb{R})$.

2. Let $u = (2, -1, 0)$, a vector in $\mathbb{R}^3$. Compute $\|u\|_1, \|u\|_2, \|u\|_3, \|u\|_\infty$.

   $\|u\|_1 = |2| + |-1| + |0| = 3$. $\|u\|_2 = \sqrt{2^2 + (-1)^2 + 0^2} = \sqrt{5}$.
   $\|u\|_3 = \sqrt[3]{|2|^3 + |-1|^3 + |0|^3} = \sqrt[3]{9}$. $\|u\|_\infty = \max\{|2|, |-1|, |0|\} = 2$.

3. Consider the vector space $M_{2,2}$ with the usual inner product $\langle X, Y\rangle = tr(X^T Y)$. Set $A = \left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$, and $B = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$. Find B_1, B_2 such that $B = B_1 + B_2$, B_1 is a multiple of A, and B_2 is orthogonal to A.

   We set $B_1 = proj(B, A) = \frac{\langle A,B\rangle}{\langle A,A\rangle} A = \frac{4}{10} A = \left[\begin{smallmatrix} 0.4 & 0.8 \\ 0.8 & 0.4 \end{smallmatrix}\right]$. We now find $B_2 = B - B_1 = \left[\begin{smallmatrix} 0.6 & -0.8 \\ 0.2 & 0.6 \end{smallmatrix}\right]$. If you wanted, you could double-check that $\langle A, B_2\rangle = 0$.

The remaining two questions concern vector space $V = P_1(x)[0, 1]$, the set of polynomials of degree at most 1 on interval $[0, 1]$ with inner product $\langle u(x), v(x)\rangle = \int_0^1 u(x)v(x)dx$. Let $S = \{s_1, s_2\}$ for $s_1(x) = \sqrt{3}x$, $s_2(x) = -3x + 2$.

4. Prove that S is an orthonormal set (hence a basis).

   We need to show three things: $\langle s_1, s_1\rangle = 1$, $\langle s_2, s_2\rangle = 1$, $\langle s_1, s_2\rangle = 0$. We have $\langle s_1, s_1\rangle = \int_0^1 3x^2 dx = x^3|_0^1 = 1$. We have $\langle s_2, s_2\rangle = \int_0^1 (-3x + 2)^2 dx = \int_0^1 (9x^2 - 12x + 4)dx = 3x^3 - 6x^2 + 4x|_0^1 = 1$. Lastly, we have $\langle s_1, s_2\rangle = \int_0^1 (-3x + 2)(\sqrt{3}x)dx = \sqrt{3}\int_0^1 (-3x^2 + 2x)dx = \sqrt{3}(-x^3 + x^2)|_0^1 = 0$.

5. Find the Fourier coefficients of $u(x) = x + 1$ with respect to S.

   Recall that the Fourier coefficients are the coefficients in the decomposition $u = \langle u, s_1\rangle s_1 + \langle u, s_2\rangle s_2$. We have $\langle u, s_1\rangle = \int_0^1 (x + 1)\sqrt{3}x dx = \sqrt{3}\int_0^1 (x^2 + x)dx = \sqrt{3}(\frac{x^3}{3} + \frac{x^2}{2})|_0^1 = \frac{5\sqrt{3}}{6}$. We also have $\langle u, s_2\rangle = \int_0^1 (x+1)(-3x+2)dx = \int_0^1 (-3x^2 - x + 2)dx = -x^3 - \frac{x^2}{2} + 2x|_0^1 = \frac{1}{2}$.