Math 254 Fall 2012 Exam 4 Solutions

1. Carefully state the definition of "basis". Give two examples from $\mathbb{R}^2$.

   SOLUTION 1: A basis is a set of vectors that is independent and spanning.
   SOLUTION 2: A basis is a maximal set of vectors that is independent.
   SOLUTION 3: A basis is a minimal set of vectors that is spanning.
   Many examples are possible, e.g. $\{(1,0),(0,1)\}, \{(1,1),(1,-1)\}$.

2. Carefully state exactly five of the eight vector space axioms.

   You may find the list on p.152 of your text. To receive full credit, these must be written carefully. e.g. "$u+v=v+u$" is not correct, and neither is "for all u,v, $u+v=v+u$."

3. Let $u=(1,2), v=(1,-2), w=(1,1)$. Determine whether w is in $Span(u,v)$.

   SOLUTION 1: We first put $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ into row canonical form, via $\left[\begin{smallmatrix} 1 & 2 \\ 1 & -2 \end{smallmatrix}\right]$ ${}_{R_2 \to R_2 - R_1}$: $\left[\begin{smallmatrix} 1 & 2 \\ 0 & -4 \end{smallmatrix}\right]$ ${}_{R_2 \to (-0.25)R_2}$: $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$ ${}_{R_1 \to R_1 - 2R_2}$: $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$. We now put $\left[\begin{smallmatrix} u \\ v \\ w \end{smallmatrix}\right]$ into row canonical form, via $\left[\begin{smallmatrix} 1 & 2 \\ 1 & -2 \\ 1 & 1 \end{smallmatrix}\right]$ ${}^{R_2 \to R_2 - R_1}_{R_3 \to R_3 - R_1}$: $\left[\begin{smallmatrix} 1 & 2 \\ 0 & -4 \\ 0 & -1 \end{smallmatrix}\right]$ ${}_{R_2 \to (-0.25)R_2}$: $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \\ 0 & -1 \end{smallmatrix}\right]$ ${}^{R_1 \to R_1 - 2R_2}_{R_3 \to R_3 + R_2}$: $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$. This has the same rowspace as $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$, so $Span(u,v,w) = Span(u,v)$ and hence w is indeed in $Span(u,v)$.

   SOLUTION 2: We first put $\left[\begin{smallmatrix} u \\ v \end{smallmatrix}\right]$ into row canonical form, as in solution 1. This rowspace is all of $\mathbb{R}^2$, hence every vector (including w) is in $Span(u,v)$.

   SOLUTION 3: We observe that $0.75u + 0.25v = (0.75, 1.5) + (0.25, -0.5) = (1,1) = w$, so w is a linear combination of u,v and hence is in $Span(u,v)$.

For problems 4,5 let $S = \{f(x) : f(0) = f(1)\} \subseteq P_2(x)$ be the set of all polynomials $f(x)$ of degree at most 2 satisfying $f(0) = f(1)$.

4. Prove that S is a vector space.

   First, the eight vector space axioms are inherited from $P_2(x)$, so we only need to check closure. Suppose $f(x), g(x)$ are in S. We have $(f+g)(0) = f(0) + g(0) = f(1) + g(1) = (f+g)(1)$, so $(f+g)(x)$ is in S. Now let c be a constant. $(cf)(0) = cf(0) = cf(1) = (cf)(1)$, so $(cf)(x)$ is in S. Hence S is closed under vector addition and scalar multiplication.

5. Let $T_1 = Span(\{x^2 - x\}), T_2 = Span(\{1\})$. Prove that $S = T_1 \oplus T_2$.

   We need to prove that $T_1 \cap T_2 = \{0\}$, and that $T_1 + T_2 = S$. Choose any vector $f(x)$ in both T_1 and T_2. Then there must be some numbers a, b where $f(x) = a(x^2 - x) = b(1)$; hence $a = b = 0$ and $f(x) = 0$. This proves the first part. For the second part, there are two approaches.

   SOLUTION 1: Let $f(x) = r_0 + r_1 x + r_2 x^2$ be in S. $f(0) = r_0, f(1) = r_0 + r_1 + r_2$, hence $r_1 + r_2 = 0$ or $r_1 = -r_2$ and in fact $f(x) = r_0 - r_2 x + r_2 x^2$. But now we can write $f(x) = r_2(x^2 - x) + r_0(1)$, the sum of a vector from T_1 and a vector from T_2. This proves the second part.

   SOLUTION 2: We have $T_1 + T_2 \subseteq S$, since both $x^2 - x, 1$ are in S. The dimension of T_1, T_2 are each 1, so the dimension of $T_1 + T_2$ is at least 1. But it can't be 1 since $T_1 \neq T_2$, so it's at least 2. On the other hand, the dimension of S is at most 3 since it's in $P_2(x)$, but it can't be 3 since $S \neq P_3(x)$, so the dimension of S is at most 2. But now $T_1 + T_2, S$ each have dimension 2 so they must in fact be equal.