Math 254 Fall 2012 Exam 2a Solutions

1. Carefully state the definition of “degenerate” in the context of linear combinations. Give two examples.

A linear combination is degenerate if all its coefficients are zero. Many examples are possible, such as 0, 0x + 0y, 0x + 0x2 + 0x3.

2. Solve the following system, using back-substitution. Be sure to justify your calculations.

\[
\begin{align*}
4x_1 + 3x_2 + 2x_3 + x_4 &= 6 \\
4x_2 - 2x_3 + x_4 &= 10 \\
5x_3 + 5x_4 &= 5 \\
2x_4 &= 8
\end{align*}
\]

\[2x_4 = 8, \text{ hence } x_4 = 4.\]
\[5x_3 + 20 = 5, \text{ hence } 5x_3 = -15 \text{ and } x_3 = -3.\]

\[4x_1 + 0 - 6 + 4 = 6, \text{ hence } 4x_1 = 8 \text{ and } x_1 = 2.\]
Combining, there is a unique solution \((x_1, x_2, x_3, x_4) = (2, 0, -3, 4)\).

3. Consider the system of equations \(\{3x - 2y = 1, kx + 4y = -2\}\). For which values of \(k\) (if any) does this have exactly one solution (and what is it)? For which values of \(k\) (if any) does this have no solution? For which values of \(k\) (if any) does this have infinitely many solutions?

Replacing Eq.1 by \((2 \text{ Eq.1} + \text{Eq.2})\), we get \((6 + k)x + 0y = 0\). If \(k \neq -6\), then this implies \(x = 0\); substituting into Eq.2 we get \(4y = -2\) and \(y = -0.5\). So, for \(k \neq -6\), there is exactly one solution: \((0, -0.5)\). However, if \(k = -6\), then we get infinitely many solutions: \((x, \frac{3x-1}{2})\), for every \(x\). ‘No solutions’ cannot occur, since we have considered all possible values for \(k\).

4. Find the line of best fit for the following set of points: \(\{(2, 0), (1, -1), (0, 4)\}\).

We calculate \(N = 3, \sum x_i = 2 + 1 + 0 = 3, \sum x_i^2 = 4 + 1 + 0 = 5, \sum y_i = 0 - 1 + 0 = -1\). Our system to solve is \(\{3b + 3m = 3, 3b + 5m = -1\}\). Eq.2-Eq.1 is \(2m = -4\), so \(m = -2\). Plugging into Eq.1 we get \(3b - 6 = 3\), so \(b = 3\). Hence \(y = -2x + 3\) is the desired line.

5. Solve the following system of linear equations using Gaussian elimination and back-substitution.

\[
\begin{align*}
2x + y + 2z &= 1 \\
-4x + 3z &= 1 \\
6x - 2y - 3z &= 4
\end{align*}
\]

We begin with Eq.2-\(\text{Eq.2} + 2 \text{ Eq.1}\), Eq.3-\(\text{Eq.3} - 3 \text{ Eq.1}\). This gives \(2x + y + 2z = 1\) \(0 + 2y + 7z = 3\). We now do Eq.3-\(2 \text{ Eq.3} + 5 \text{ Eq.2}\), to get \(0 + 2y + 7z = 3\). Now we can do back-substitution: \(17z = 17\), so \(z = 1\); \(2y + 7 = 3\), so \(y = -2\); \(2x - 2 + 2 = 1\), so \(x = 0.5\). Hence there is a unique solution \((x, y, z) = (0.5, -2, 1)\).