Math 254 Fall 2012 Exam 10 Solutions

1. Carefully state the definition of "basis". Give two examples for $P_2(t)$.

   (1) A basis is a set of vectors that is independent and spanning; or (2) A basis is a maximal set of vectors that is independent; or (3) A basis is a minimal set of vectors that is spanning. Many examples are possible: the standard basis is $\{1, t, t^2\}$, but also $\{1, t+1, t^2+1\}$ and $\{t+1, t^2+1, 3t^2+t+1\}$.

2. Recall that $M_{2,2}$ denotes the vector space of all 2×2 matrices. Prove or find a counterexample to the following: For all $A, B \in M_{2,2}$, $|A+B| = |A| + |B|$.

   The statement is false, so we need a counterexample. These are plentiful, such as $A = B = I_2$. We have $|A+B| = 4$ but $|A| + |B| = 2$.

The remaining three problems all concern the matrix $A = \begin{bmatrix} 2 & 2 & 2 \\ -1 & 0 & 2 \\ 4 & 2 & 0 \end{bmatrix}$.

3. Compute $|A|$ directly, using either determinant formula.

   We write $\begin{bmatrix} 2 & 2 & 2 & 2 & 2 \\ -1 & 0 & 2 & -1 & 0 \\ 4 & 2 & 0 & 4 & 2 \end{bmatrix}$, and calculate $|A| = (2)(0)(0)+(2)(2)(4)+(2)(-1)(2)-(4)(0)(2)-(2)(2)(2)-(0)(-1)(2) = 0+16-4-0-8-0 = 4$.

4. Compute $|A|$ by finding the Laplace expansion of the second column.

   We have $|A| = (-1)^{1+2}(2)\begin{vmatrix} -1 & 2 \\ 4 & 0 \end{vmatrix} + (-1)^{2+2}(0)\begin{vmatrix} 2 & 2 \\ 4 & 0 \end{vmatrix} + (-1)^{3+2}(2)\begin{vmatrix} 2 & 2 \\ -1 & 2 \end{vmatrix} = -2(-8)+0(-8)-2(6) = 16-12 = 4$.

5. Compute $|A|$ by first making A triangular with elementary row operations.

   Many solutions are possible. For example, $\begin{bmatrix} 2 & 2 & 2 \\ -1 & 0 & 2 \\ 4 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 6 \\ -1 & 0 & 2 \\ 0 & 2 & 8 \end{bmatrix}$ via $R_1 \to R_1 + 2R_2$, $R_3 \to R_3 + 4R_2$, both of which leave the determinant unchanged. Then, $\begin{bmatrix} 0 & 2 & 6 \\ -1 & 0 & 2 \\ 0 & 2 & 8 \end{bmatrix} \to \begin{bmatrix} 0 & 2 & 6 \\ -1 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix}$ via $R_3 \to R_3 - R_1$, which leaves the determinant unchanged. Lastly, $\begin{bmatrix} 0 & 2 & 6 \\ -1 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} -1 & 0 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 2 \end{bmatrix}$ via $R_1 \leftrightarrow R_2$, which reverses the sign of the determinant. This last matrix has determinant $(-1)(2)(2) = -4$, so because we changed signs $|A| = 4$.