Math 254 Fall 2012 Exam 1 Solutions

1. Carefully state the definition of "spanning". Give two examples in $\mathbb{R}^2$.

   A set of vectors is spanning if every vector may be obtained as a linear combination of this set. Many examples are possible, such as $\{(1,0),(0,1)\}$, $\{(1,0),(0,1),(1,1)\}$, $\{(1,0),(1,1)\}$, $\{(1,0),(0,1),(0,0)\}$. All correct examples are (correctly chosen) sets of vectors from $\mathbb{R}^2$.

2. Let $u = \begin{bmatrix} 1 & 2 & -67 & 89 \end{bmatrix}$, $v = \begin{bmatrix} 4 & 5 & 6 & 7 \end{bmatrix}$. For each of the following, determine what *type* they are (undefined, scalar, matrix/vector). If a matrix/vector, specify the dimensions.

   uv^T: 1×1 matrix, or a scalar
   $u^T v$: 4×4 matrix
   $(u^T + v^T)^T$: 1×4 matrix, or a row vector
   $(u \cdot v) \cdot u$: undefined
   $(u \times v) \times u$: undefined

3. Let $u = (-2,-3,-4)$, $v = (1,2,-2)$. Determine, with justification, whether these vectors (in $\mathbb{R}^3$) are orthogonal.

   $u \cdot v = -2 - 6 + 8 = 0$, so u, v are orthogonal.

4. Let $A = \begin{bmatrix} 0 & 1 & 1 \\ -1 & -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 4 \\ 0 & -1 \\ -1 & 2 \end{bmatrix}$. Calculate AB and BA.

   $AB = \begin{bmatrix} -1 & 1 \\ -5 & 1 \end{bmatrix}$. $BA = \begin{bmatrix} -4 & -1 & 11 \\ 1 & 1 & -2 \\ -2 & -3 & 3 \end{bmatrix}$.

5. Let $u = (1,-1,2)$, $v = (0,3,1)$. Calculate $u \times v$ and $v \cdot (u \times v)$.

   We first calculate $u \times v$.
   Method 1: $(\hat{\imath} - \hat{\jmath} + 2\hat{k}) \times (3\hat{\jmath} + \hat{k}) = 3(\hat{\imath} \times \hat{\jmath}) + (\hat{\imath} \times \hat{k}) - 3(\hat{\jmath} \times \hat{\jmath}) - (\hat{\jmath} \times \hat{k}) + 6(\hat{k} \times \hat{\jmath}) + 2(\hat{k} \times \hat{k}) = 3\hat{k} - \hat{\jmath} - \hat{\imath} - 6\hat{\imath} = (-7,-1,3)$.
   Method 2: $\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = \hat{\imath}(-1-6) - \hat{\jmath}(1-0) + \hat{k}(3-0) = (-7,-1,3)$.
   We now calculate $v \cdot (u \times v)$.
   Method 1: $v \cdot (u \times v) = (0,3,1) \cdot (-7,-1,3) = 0 - 3 + 3 = 0$.
   Method 2: This is the triple product, which gives the (signed) volume of the parallelepiped formed by the vectors v, u, v. However this figure is two-dimensional and has no volume; hence the triple product equals zero.