Math 254 Fall 2012 Exam 0 Solutions

1. Carefully state the definition of "linear equation". Give two examples.

   A linear equation is equivalent to one where a linear combination is set equal to a constant. Many examples are possible, such as $3x = 4, 7x - 2y = 3, 0 = 2, 0 = 0$.

2. Carefully state the definition of "dependent". Give two examples in $\mathbb{R}^2$.

   A set of vectors is dependent if there is a *nondegenerate* linear combination yielding the zero vector. Many examples are possible, such as $\{(0,0)\}$, $\{(1,0),(2,0)\}$, $\{(1,2),(2,4)\}$, $\{(1,0),(0,1),(3,4)\}$. All correct examples must be sets containing vectors.

3. Consider $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f((x_1, x_2, x_3)) = (1, x_2 + x_3)$. Prove that this is *not* a linear transformation.

   This requires finding a counterexample to either of the two linear transformation properties. Such counterexamples abound, since it turns out that neither property EVER holds. e.g., let $x = (1,2,3)$. Then $f(7x) = f((7,14,21)) = (1,35)$ but $7f(x) = 7(1,5) = (7,35)$. Because $(1,35) \neq (7,35)$ this is not linear. Or, instead we could try $f(x+x) = f((2,4,6)) = (1,10)$ but $f(x)+f(x) = (1,5)+(1,5) = (2,10)$. Because $(1,10) \neq (2,10)$ this is not linear.

4. Consider the set S of all $\overline{v} = (v_1, v_2)$ such that $v_1 = 2v_2$. This is a subset of the vector space $\mathbb{R}^2$. Prove that S is in fact a subspace of $\mathbb{R}^2$.

   We must prove that S is closed under vector addition and scalar multiplication. Let $\overline{v}, \overline{w}$ be arbitrary vectors in S. Then $v_1 = 2v_2, w_1 = 2w_2$. Adding, $v_1 + w_1 = 2v_2 + 2w_2 = 2(v_2 + w_2)$. Hence $(\overline{v} + \overline{w}) = (v_1 + w_1, v_2 + w_2)$ is in S. Now, let a be an arbitrary scalar, and again $\overline{v}$ an arbitrary vector in S. Then $v_1 = 2v_2$; multiplying by a we get $av_1 = a(2v_2) = 2(av_2)$. Hence $a\overline{v} = (av_1, av_2)$ is in S.

5. Consider the vector space $\mathbb{R}^2$. Show that the following set is independent: $\{(2,6),(0,-9)\}$.

   We must prove that ONLY a degenerate linear combination yields zero. Suppose that we had a linear combination yielding zero; then we have some constants a, b with $a(2,6) + b(0,-9) = (0,0)$. We simplify as $(2a, 6a - 9b) = (0,0)$. Hence both $2a = 0$ and $6a - 9b = 0$ must hold. But the first equation implies $a = 0$; plugging into the second we get $b = 0$ as well. Hence the linear combination was degenerate all along.

   ALTERNATE SOLUTION: If this set were dependent, then its span would be either a zero-dimensional or one-dimensional subspace. Since its span contains a nonzero element, it is not zero-dimensional. If its span were one-dimensional, then every nonzero element would be a multiple of every other nonzero element. Hence $(2,6) = a(0,-9)$ for some scalar a. However no a makes this equal (the first coordinate is always zero), so the span is not one-dimensional. NOTE: this fact is only true about one-dimensional subspaces.