MATH 254: Introduction to Linear Algebra
Chapter 0: Fundamental Definitions of Linear Algebra

This course contains ten fundamental definitions which will be tested repeatedly. Please *memorize* them, as well as how to apply them to specific examples. They will be tested on every exam.

1. A **linear function/combination*** is a function, in one or more variables, that is entirely some mixture of addition and multiplication by constants, AND no constant is added by itself.

Examples: $f(x,y) = 3x + 2y$, $g(a,b,c,d) = 2a - b + 0c + 3d$, $f(x,y) = 0x + 0y$, $3z$, $7z + 8w$
Non-examples: $f(x,y) = 3x + 2y + 2$, $f(x) = x^2$, $f(x,y) = xy + y$, $f(x) = e^x$, $\sin x$
Note: the "no constant is added by itself" condition is equivalent to $f(0,0,\ldots,0) = 0$.

2. A **linear equation** is an equation setting a linear combination equal to any constant.
Examples: $3x + 2y = 2$, $2a - b + 0c + 3d = 0$, $3z = 12$, $x_1 + x_2 - x_3 = -4$, $0x + 0y = 3$, $0x + 0y = 0$
Non-examples: $x^2 = 2$, $xy + y = 7$, $x + \sin(y) = 4$, $e^x = 0$

3. A linear equation/function/combination is called **nondegenerate** if it contains at least one variable with a coefficient that is nonzero. It is **degenerate** if the coefficient of every variable is zero.

Nondegenerate: $3x + 2y = 2$, $2a - b + 0c + 3d = 0$, $f(z) = 3z$, $g(x_1, x_2) = 7x_1 + 0x_2$, $7x + 3y$
Degenerate: $0x + 0y = 2$, $0a + 0b + 0c + 0d = 0$, $f(x,y) = 0x + 0y$, $0x - 2y$, 0

4. A **vector space** is a collection of objects (called vectors) and numbers (called scalars), that satisfy a certain list of properties. The most important of these properties is *closure*: every linear combination of vectors in the vector space, must again be a vector in the vector space. We will study the other properties later in the course. In this course, the scalars are almost always the real numbers $\mathbb{R}$.

One may USE closure as any linear combination (e.g. $5u + 7v - 8w$ is again a vector). To PROVE closure is easier: one needs to prove only two specific types of linear combination, not all of them:

- For every vector v and every scalar a, av is a vector, "scalar multiplication"
- For every two vectors u, v their sum $u + v$ is a vector. "vector addition"

Important Example 1: $\mathbb{R}^2$ is ordered pairs of real numbers (the vectors), such as $u = (1,2), v = (0,-1)$. A linear combination looks like $-2u + 4v = -2(1,2) + 4(0,-1) = (-2,-8)$, which is again in $\mathbb{R}^2$.
Important Example 2: $\mathbb{R}^3$ is ordered triples of real numbers, such as $(1,2,3)$.
Important Example 3: $\mathbb{R}^n$ is ordered lists of n real numbers.
Non-Example: Ordered pairs (a,b) where $a > b$ are NOT a vector space; although closed under vector addition, they are not closed under scalar multiplication: $-2(5,4) = (-10,-8)$, which does not satisfy $-10 > -8$.

Notation: A vector in $\mathbb{R}^n$ can be written as a list of n numbers, using subscripts, between parentheses. For example, if u is a vector in $\mathbb{R}^3$, then we may write $u = (u_1, u_2, u_3)$. A set is always denoted with curly braces, e.g. $\{u\} = \{(u_1, u_2, u_3)\}$ is the set containing the single vector u.

5. A **linear mapping/transformation** is a function, in one variable, from (the vectors of) a vector space to another (possibly the same) vector space. This function f must satisfy two properties:

- For every vector v and every scalar a, $f(av) = af(v)$, and
- For every two vectors u, v, $f(u + v) = f(u) + f(v)$.

*The difference between "function" and "combination" is that a combination is a function with no specified name.

Examples: $f(x) = 2x$, rotation, stretching, matrix multiplication, differentiation
Non-examples: $f(x) = \sin(x)$, because $\sin(\pi/2 + \pi/2) = \sin(\pi) = 0 \neq 2 = \sin(\pi/2) + \sin(\pi/2)$.
$f(x) = e^x$, because $e^{0+0} = e^0 = 1 \neq 2 = e^0 + e^0$. $f(x) = x+3$, because $(7+7)+3 = 17 \neq 20 = (7+3)+(7+3)$.

6. A **subspace** of a vector space is itself a vector space, contained within a bigger one. Apart from closure, all of the other vector space properties are all inherited, for free, from the larger vector space.

Important Example 1: The set of all linear combinations of any set of vectors. (the "span" of this set)
Important Example 2: The range of any linear mapping. (the "image" of this mapping)
Important Example 3: The set of vectors that a linear mapping sends to 0. (the "kernel" of this mapping)
Example 4: Consider the set S of all $v = (v_1, v_2)$, where $v_1 + v_2 = 0$. One could show this is a subspace directly, by checking closure. Or, observe that S is the kernel of the mapping $f : \mathbb{R}^2 \to \mathbb{R}^1$ given by $f((x_1, x_2)) = (x_1 + x_2)$. Now prove that f is a linear mapping. In either case, S is a subspace of $\mathbb{R}^2$.
Example 5: Consider the set T of all $v = (v_1, 0)$, another subset of $\mathbb{R}^2$. Again one could show that T is a subspace of $\mathbb{R}^2$ directly, or indirectly by considering $f : \mathbb{R}^2 \to \mathbb{R}^1$ given by $f((x_1, x_2)) = (x_1)$.
Non-example: Consider the set U of all $v = (v_1, 7)$, yet another subset of $\mathbb{R}^2$. This is not closed because $2v = (2v_1, 14)$ is not in U. The first closure property fails for $a = 2$ (and so is not true for all a and all v).

7. A set of vectors is called **dependent** if there is a nondegenerate linear combination of those vectors that yields the zero vector. Otherwise, the set of vectors is called **independent** – if EVERY nondegenerate linear combination of that set yields a NONzero vector.

Dependent: $x = (1, 1), y = (1, 2), z = (1, 3)$. $x - 2y + z = (1, 1) - 2(1, 2) + (1, 3) = (0, 0)$.
Independent: $x = (1, 1), y = (0, 1)$. Consider any linear combination $ax + by = a(1, 1) + b(0, 1) = (a, a + b)$. If this equals $(0, 0)$, then $a = b = 0$. Hence the zero vector cannot be expressed by a nondegenerate linear combination on x, y.

8. A set of vectors is called **spanning** if every vector in the vector space can be expressed as a linear combination of the elements of that set.

Example 1: $\{(1, 0), (0, 1)\}$ spans $\mathbb{R}^2$. For every v in $\mathbb{R}^2$, $v = (v_1, v_2) = v_1(1, 0) + v_2(0, 1)$.
Example 2: $\{(1, 0), (0, 1), (0, 4)\}$ spans $\mathbb{R}^2$. For every v in $\mathbb{R}^2$, $v = (v_1, v_2) = v_1(1, 0) + v_2(0, 1) + 0(0, 4)$.
Example 3: $\{(1, 1), (0, 1)\}$ spans $\mathbb{R}^2$. For every v in $\mathbb{R}^2$, $v = (v_1, v_2) = v_1(1, 1) + (v_2 - v_1)(0, 1)$.
Non-example: $\{(1, 1)\}$ does not span $\mathbb{R}^2$. The linear combinations of this set all have their two coordinates equal, and so $(2, 3)$, to pick just one example, cannot be expressed in this way.

9. A **basis** is a set of vectors that is both spanning and independent. Equivalently, it is a maximal set of independent vectors ("maximally independent"). Equivalently, it is a minimal set of spanning vectors ("minimally spanning"). Typically a vector space has many bases, but we like some more than others.

Important Example 1: The standard basis for $\mathbb{R}^n$ is $\{e_1, e_2, \ldots, e_n\}$, where e_i has a 1 in the i^{th} position and 0 in every other position. e.g. $\mathbb{R}^3$ has basis $\{e_1, e_2, e_3\} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$.
Example 2: $\{(1, 1), (0, 1)\}$ was shown above to be both independent and spanning. Thus it is a basis for $\mathbb{R}^2$.

10. The **dimension** of a vector space is the number of elements of a basis. It turns out that this number is the same, no matter what basis you choose.

Important Example 1: $\mathbb{R}^n$ has dimension n.
Important Example 2: The set of all linear combinations of k linearly independent vectors has dimension k.
Example 3: Consider the set of all $v = (v_1, v_2)$ where $v_1 + v_2 = 0$. This is a subspace of $\mathbb{R}^2$. Its dimension cannot be 2 (since it is not all of $\mathbb{R}^2$), nor 0 (since it contains nonzero vectors), hence must be 1. It has basis $\{(3, -3)\}$.

Solved Problems

1. Carefully state the definition of "Linear Function".

   A linear function is a function that combines addition and multiplication by constants ONLY, and no constant is added by itself.

2. Carefully state the definition of "Degenerate Function".

   A linear function is degenerate if it is always zero no matter what it is evaluated on.

3. Carefully state the definition of "Linear Transformation".

   A linear transformation is a one-variable vector function f that satisfies $f(av) = af(v)$ and $f(u + v) = f(u) + f(v)$ (for every scalar a, and all vectors u, v).

4. Carefully state the definition of "Subspace".

   A subspace is a vector space that is contained within a larger vector space.

5. Carefully state the definition of "Basis".

   Solution 1: A basis is a set of vectors that is both spanning and independent.
   Solution 2: A basis is a maximal set of independent vectors.
   Solution 3: A basis is a minimal set of spanning vectors.

6. Determine which of the following equations is linear (justify your answers). Which are degenerate? A: $0x + 3y = 2y - 7$, B: $0x + 0y + 0z = 7$, C: $3x + 0xy = 7y$, D: $x/y = 3$

   A: This equation is equivalent to $0x + 1y = -7$ (or $1y = -7$), a linear combination set equal to a constant. Hence, it is linear. It is nondegenerate since the coefficient of y is 1.
   B: This equation is a linear combination set equal to a constant. Hence, it is linear, but degenerate. There are no (x, y, z) that satisfy this equation, incidentally.
   C: The equation is equivalent to $3x - 7y = 0$, hence it is linear, and nondegenerate. (the $0xy$ term does not ruin the linearity because of the 0).
   D: The x/y term DOES ruin the linearity. Although it is possible to multiply both sides by y to get $x = 3y$, a linear equation, the two equations are not equivalent. $x = y = 0$ satisfies $x = 3y$, but does not satisfy $x/y = 3$, so the two equations are (subtly) different. Since it is not linear, it is not meaningful to ask whether it is degenerate.

7. Consider the vector space $\mathbb{R}^3$, and set $v = (-3, 2, 0), u = (0, 1, 4)$. Calculate $2v - u$.

   $2v - u = 2(-3, 2, 0) - (0, 1, 4) = (-6, 4, 0) + (0, -1, -4) = (-6, 3, -4)$

8. Consider the vector space $\mathbb{R}^2$ and the vector function $f(x) = f((x_1, x_2)) = (2x_2, x_1)$. Determine whether or not f is a linear transformation.

   We calculate $f(ax) = f((ax_1, ax_2)) = (2ax_2, ax_1) = a(2x_2, x_1) = af(x)$. Since a, x were arbitrary, $f(ax) = af(x)$ holds for every scalar a and every vector x. We now calculate $f(x+y) = f((x_1+y_1, x_2+y_2)) = (2(x_2+y_2), (x_1+y_1)) = (2x_2+2y_2, x_1+y_1) = (2x_2, x_1) + (2y_2, y_1) = f(x)+f(y)$. Since x, y were arbitrary, $f(x+y) = f(x)+f(y)$ holds for all vectors x, y. Hence f is a linear transformation.

9. Consider the vector space $\mathbb{R}^2$ and the vector function $f(x) = f((x_1, x_2)) = (x_1x_2, 0)$. Determine whether or not f is a linear transformation.

   We calculate $f(ax) = f((ax_1, ax_2)) = (ax_1ax_2, 0) = a^2(x_1x_2, 0)$. This doesn't look like $af(x)$, but that's not sufficient – appearances can be deceptive. We must find some specific a, x that are a counterexample. Pulling some numbers out of my ear, let's try $a = 2, x = (1, 1)$. $f(ax) = f((2, 2)) = (4, 0)$, whereas $af(x) = 2f((1, 1)) = 2(1, 0) = (2, 0)$. Hence f is NOT a linear transformation.

10. Consider the set S of all $v = (v_1, v_2)$ such that $|v_1| \geq |v_2|$. This is a subset of $\mathbb{R}^2$. Is it a subspace?

    For any scalar a and any vector v in S, we calculate $av = a(v_1, v_2) = (av_1, av_2)$. Because

$|v_1| \geq |v_2|$, we may multiply both sides by the nonnegative $|a|$ to get $|a||v_1| \geq |a||v_2|$ and hence $|av_1| \geq |av_2|$. Hence av is a vector in S; the first closure property holds.
We now take two vectors u, v in S, and calculate $u+v = (u_1, u_2)+(v_1, v_2) = (u_1+v_1, u_2+v_2)$. Must $|u_1 + v_1| \geq |u_2 + v_2|$? Perhaps not, so we need to find a specific counterexample. Many are possible, for example $u = (3, 1), v = (-3, 1)$. Both of u, v are in S, but $u + v = (0, 2)$ is not. Hence the second closure property does NOT hold. S is not a subspace, since to be a subspace both closure properties must hold.

11. Consider the vector space $\mathbb{R}^2$ and the linear transformation $f(x) = f((x_1, x_2)) = (2x_2, x_1)$. Consider the set of vectors that f sends to 0 (the "kernel" of f). Find this set, and verify that it is a subspace.

    If $f(x) = 0$, then $(2x_2, x_1) = (0, 0)$ and hence $2x_2 = 0, x_1 = 0$. Hence the kernel is just the single vector $(0, 0)$. This is indeed a subspace, since $a(0, 0) = (0, 0)$ and $(0, 0)+(0, 0) = (0, 0)$ (hence it is closed).

12. Consider the vector space $\mathbb{R}^2$ and the linear transformation $f(x) = f((x_1, x_2)) = (x_1+x_2, 0)$. Consider the set of vectors that f sends to 0 (the "kernel" of f). Find this set, and verify that it is a subspace.

    If $f(x) = 0$, then $(x_1 + x_2, 0) = (0, 0)$, and hence $x_1 + x_2 = 0$. Hence the kernel is all vectors (x_1, x_2) such that $x_1+x_2 = 0$. To verify it is a subspace, we must verify closure. We calculate $ax = a(x_1, x_2) = (ax_1, ax_2)$. Since $x_1 + x_2 = 0$, we must also have $ax_1 + ax_2 = 0$. Hence if x is in the kernel, ax is as well for every scalar a (first closure property). We calculate $x + y = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$. Since $x_1 + x_2 = 0$ and $y_1 + y_2 = 0$, in fact $x_1+y_1+x_2+y_2 = 0$. Hence if x, y are in the kernel, $x+y$ is a well (second closure property). Since both closure properties hold, this is indeed a subspace.

13. Consider the vector space $\mathbb{R}^2$, and set $u = (1, 1), v = (2, 3), w = (0, 5)$. Determine whether or not $\{u, v, w\}$ is dependent (justify your answer).

    Solution 1: To prove that $\{u, v, w\}$ is dependent, we need to find a nondegenerate linear combination yielding zero. Consider $10u-5v+w$, found by a side calculation. $10u-5v+w = 10(1, 1) - 5(2, 3) + (0, 5) = (10, 10) - (10, 15) + (0, 5) = (0, 0)$. Hence, $\{u, v, w\}$ is dependent.
    Solution 2: If $\{u, v, w\}$ were independent, then $\mathbb{R}^2$ would have dimension at least three; however $\mathbb{R}^2$ has dimension 2. Therefore, $\{u, v, w\}$ must be dependent.

14. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 2), v = (3, 0)$. Determine whether or not $\{u, v\}$ is dependent (justify your answer).

    Solution 1: To prove that $\{u, v\}$ is independent, we need to prove that any nondegenerate linear combination does not yield the zero vector. Suppose, to the contrary, that there *were* such a linear combination, i.e. some constants a, b (not both zero) so that $au + bv = (0, 0)$. We calculate $au + bv = a(2, 2) + b(3, 0) = (2a, 2a) + (3b, 0) = (2a + 3b, 2a) = (0, 0)$. So, we must have $2a + 3b = 0$ and $2a = 0$. The second equation gives us $a = 0$; we plug that into the first equation and get $b = 0$. Hence, $a = b = 0$ and the linear combination was actually degenerate (a contradiction!). Hence $\{u, v\}$ is independent.
    Solution 2: If $\{u, v\}$ were dependent, the set of all linear combinations would be a subspace of dimension 1. In this case, u would be a scalar multiple of v; but it is not since all scalar multiples of v have a 0 in the second coordinate. WARNING: this type of solution ONLY works for subspaces of dimension 1 – compare with the next problem.

15. Consider the vector space $\mathbb{R}^3$, and set $u = (1, 1, 1), v = (-1, 0, 1), w = (1, 2, 3)$. Determine whether or not $\{u, v, w\}$ is dependent (justify your answer).

    To prove that $\{u, v, w\}$ is dependent requires a nondegenerate linear combination yielding the zero vector. We have $2u + v - w = 2(1, 1, 1) + 1(-1, 0, 1) - 1(1, 2, 3) = (2, 2, 2) + (-1, 0, 1) + (-1, -2, -3) = (0, 0, 0)$, so this set is dependent. To find this linear combination, we seek constants a, b, c (not all zero) so that $au + bv + cw = (0, 0, 0)$. We calculate $au + bv + cw = (a, a, a) + (-b, 0, b) + (c, 2c, 3c) = (a - b + c, a + 2c, a + b + 3c) = (0, 0, 0)$. Hence

$a-b+c=0, a+2c=0, a+b+3c=0$. This system has infinitely many solutions – choose c arbitrarily, then $a=-2c, b=-c$. The example above corresponded to $c=-1$.
NOTE: No one of u, v, w is a multiple of any one of the others. Although this was useful in the one-dimensional case (see the previous problem), the linear combinations of this set form a two-dimensional subspace and this approach is not helpful.

16. Consider the vector space $\mathbb{R}^2$, and set $u=(2,3)$. Determine whether or not $\{u\}$ is spanning (justify your answer).

    Solution 1: To prove that $\{u\}$ is not spanning, we must provide a counterexample. We claim that $(1,1)$ cannot be expressed as a linear combination of u, because then for some a we have $(1,1)=a(2,3)=(2a,3a)$, and hence $2a=1=3a$, which is impossible.
    Solution 2: Since $\mathbb{R}^2$ has dimension 2, any spanning set must have at least two elements. Hence this set is not spanning.

17. Consider the vector space $\mathbb{R}^2$, and set $u=(2,2), v=(3,0)$. Determine whether or not $\{u,v\}$ is spanning (justify your answer).

    Solution 1: To prove that $\{u,v\}$ is spanning, we need to prove that every vector can be expressed as a linear combination of u, v. Let $x=(x_1,x_2)$ be an arbitrary vector in $\mathbb{R}^2$. Set $a={}^{x_2}/_2$ and set $b={}^{(x_1-x_2)}/_3$ (both real numbers no matter what x is), found by a side calculation. We have $au+bv=a(2,2)+b(3,0)=(2a+3b,2a)=(x_1,x_2)=x$.
    Solution 2: By an earlier problem this set is independent. But the dimension of $\mathbb{R}^2=2$, hence in fact this set is a basis. But then it is also spanning.

18. Consider the vector space $\mathbb{R}^2$, and set $u=(2,2), v=(3,0), w=(7,5)$. Determine whether or not $\{u,v,w\}$ is spanning (justify your answer).

    To prove that $\{u,v,w\}$ is spanning, we need to prove that every vector can be expressed as a linear combination of u,v,w. Comparing with the previous problem, already every $x=au+bv$, for some real a,b. Hence $x=au+bv+0w$, a linear combination of $\{u,v,w\}$, so this set is also spanning.

19. Consider the vector space $\mathbb{R}^3$, and set $u=(1,1,1), v=(-1,0,1), w=(1,2,3)$. Determine whether or not $\{u,v,w\}$ is spanning (justify your answer).

    Solution 1: To prove that $\{u,v,w\}$ is not spanning, we must find a counterexample. We claim that $x=(1,1,0)$ is such a counterexample (found by a tricky side calculation). Suppose we could express x as a linear combination of u,v,w. Then, for some real constants a,b,c, we have $x=au+bv+cw=(a-b+c,a+2c,a+b+3c)=(1,1,0)$. Hence $a-b+c=1, a+2c=1, a+b+3c=0$. Adding the first and third equations gives $2a+4c=1$, which is inconsistent with the second equation. Hence $x=(1,1,0)$ is not expressible as a linear combination of $\{u,v,w\}$, which is therefore not spanning.
    Solution 2: The set $\{u,v,w\}$ has three elements, and $\mathbb{R}^3$ has dimension 3. Hence if this set were spanning, it would also be a basis and therefore independent. But an earlier problem showed that this set is not independent, hence it cannot be spanning.

20. Find three different bases for $\mathbb{R}^2$.

    Many solutions are possible. An easy choice is the standard basis $\{e_1,e_2\}=\{(1,0),(0,1)\}$. An earlier problem showed that $\{(2,2),(3,0)\}$ is another basis. An example given after the definition of spanning showed that $\{(1,1),(0,1)\}$ is spanning, and thus is a basis since it contains two elements.

Supplementary Problems

Be sure to thoroughly justify all your solutions.

21. Carefully state the definition of "Linear Equation".
22. Carefully state the definition of "Linear Combination".
23. Carefully state the definition of "Spanning".
24. Carefully state the definition of "Independent".
25. Carefully state the definition of "Basis".
26. For the vectors $u = (1, 2, 3), v = (4, 0, 1), w = (-3, -2, 5)$, calculate $2u - 3v - 4w$.
27. Determine which of the following functions is linear (justify your answers). A: $f(x, y) = 7x-3y+2x+4y$, B: $f(x, y) = 0x + 0y + 0$, C: $f(x, y) = 2x + 3y + 4$, D: $f(x, y, z) = (x/y)(y/z)z$, E: $f(x, y, z) = x$
28. Consider the vector space $\mathbb{R}^2$ and the vector function $f((x_1, x_2)) = (x_2, 0)$. Is this a linear mapping?
29. Consider the vector space $\mathbb{R}^2$ and the vector function $f((x_1, x_2)) = (x_1^3, 0)$. Is this a linear mapping?
30. Consider the set S of all vectors $v = (v_1, v_2)$ such that $2v_1 + v_2 = 0$. Determine whether or not this is a subspace of $\mathbb{R}^2$.
31. Consider the set S of all vectors $v = (v_1, v_2)$ such that $v_1 v_2 = 0$. Determine whether or not this is a subspace of $\mathbb{R}^2$.
32. Consider the vector space $\mathbb{R}^2$ and the linear mapping $f((x_1, x_2)) = (2x_1 + x_2, 0)$. Consider the set of vectors that f sends to 0 (the "kernel" of f). Find this set, and verify that it is a subspace.
33. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 6), v = (-3, -9)$. Determine whether or not $\{u, v\}$ is independent.
34. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 6), v = (-3, -9), w = (5, 15)$. Determine whether or not $\{u, v, w\}$ is independent.
35. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 6), v = (0, -9)$. Determine whether or not $\{u, v\}$ is independent.
36. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 6), v = (-3, -9)$. Determine whether or not $\{u, v\}$ is spanning.
37. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 6), v = (-3, -9), w = (5, 15)$. Determine whether or not $\{u, v, w\}$ is spanning.
38. Consider the vector space $\mathbb{R}^2$, and set $u = (2, 6), v = (0, -9)$. Determine whether or not $\{u, v\}$ is spanning.
39. Which of the sets given in problems 36-38 are bases of $\mathbb{R}^2$?
40. For the sets given in problems 36-38, determine the dimension of the subspace they span.

Answers to Supplementary Problems: (WARNING: these are just answers, NOT thoroughly justified solutions)

26: $(2, 12, -17)$ 27: A,B,E 28: yes 29: no 30: yes 31: no 33: no 34: no 35: yes 36: no 37: no 38: yes 39: just 38 40: 1,1,2