Math 254 Exam 8 Solutions

1. Carefully state the definition of "linear mapping". Give two examples, each from $\mathbb{R}^2$ to itself.

   A linear mapping is a function f from a vector space to a (possibly different) vector space, that satisfies $f(u+v) = f(u) + f(v)$ and $f(ku) = kf(u)$, for all scalars k and all vectors u, v. Many examples are possible, such as $f(x, y) = (x, y)$, $f(x, y) = (0, 0)$, $f(x, y) = (x, 0)$, $f(x, y) = (2x - 3y, 7x + 5y)$.

2. Consider the mapping $f : \mathbb{R}^3 \to \mathbb{R}^2$ given by $f(x, y, z) = (2z - x, x + 2y + 3z)$. Determine whether this is linear.

   First, we check that $f((x, y, z)+(x', y', z')) = f(x+x', y+y', z+z') = (2(z+z')-(x+x'), (x+x')+2(y+y')+3(z+z')) = (2z-x, x+2y+3z)+(2z'-x', x'+2y'+3z') = f(x, y, z)+f(x', y', z')$. Second, we check that $f(k(x, y, z)) = f(kx, ky, kz) = (2kz - kx, kx + 2ky + 3kz) = k(2z - x, x + 2y + 3z) = kf(x, y, z)$. Hence f is linear.

3. Consider the linear map $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by $f(x, y) = (x - 2y, 0, 4y - 2x)$. Find a basis for its kernel, and find a basis for its image.

   To find the kernel, we solve $(0, 0, 0) = f(x, y) = (x - 2y, 0, 4y - 2x)$ to get $x = 2y$. Hence the kernel is one-dimensional, with basis $\{(2, 1)\}$. By the dimension theorem, this means that the image is one-dimensional. Choosing any nonzero element of the image, say $f(1, 0) = (1, 0, -2)$, we get a basis for the image, i.e. $\{(1, 0, -2)\}$.

4. Consider the linear map $\frac{d}{dt} : P_3(t) \to P_3(t)$. Find a basis for its kernel, and find a basis for its image.

   To find the kernel, we solve $0 = \frac{d}{dt}(a + bt + ct^2 + dt^3) = b + 2ct + 3dt^2$. Hence $b = c = d = 0$, so the kernel is one-dimensional and a basis is $\{1\}$. By the dimension theorem, the image will be $4 - 1 = 3$ dimensional. To find a basis for the image, we can apply $\frac{d}{dt}$ to a basis of the domain $\{1, t, t^2, t^3\}$, yielding $\{0, 1, 2t, 3t^2\}$. Hence a basis for the image is $\{1, 2t, 3t^2\}$.

   ALTERNATE SOLUTION: $\frac{d}{dt}$ lowers the degree of every monomial by one. Hence constant monomials are killed, and all polynomials of degree at most 2 are possible as output. Hence $\{1\}$ is a basis for the kernel, and $\{1, t, t^2\}$ is a basis for the image.

5. Consider all possible linear mappings from $P_2(t)$ to $\mathbb{R}^1$. What are the possible nullities and ranks of these? Given an example for each possible combination, and indicate which are one-to-one and which are onto.

   Since $P_2(t)$ is 3-dimensional, the sum of nullity+rank must equal 3. However, since $\mathbb{R}^1$ is one-dimensional, the rank can be either 0 or 1; hence the nullity is either 3 or 2, respectively. Many examples are possible, such as:
   $f(a + bt + ct^2) = (0)$ nullity=3, rank=0, not one-to-one, not onto
   $f(a + bt + ct^2) = (b + 2c)$ nullity=2, rank=1, not one-to-one, onto