Math 254 Exam 7 Solutions

1. Carefully state the definition of "linearly dependent". Give two examples, each from $M_{2,2}$.

   A set of vectors is linearly dependent if there is some nondegenerate linear combination yielding the zero vector. Many examples are possible, such as $\{\left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)\}$, $\{\left(\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}2&2\\2&0\end{smallmatrix}\right)\}$, $\{\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right), \left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right), \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right)\}$. (note that $\left(\begin{smallmatrix}1&1\\0&0\end{smallmatrix}\right) - \left(\begin{smallmatrix}1&0\\1&0\end{smallmatrix}\right) + \left(\begin{smallmatrix}0&-1\\1&0\end{smallmatrix}\right) = \left(\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right)$.)

2. Let $u = (0, -4, 3)$, a vector in $\mathbb{R}^3$. Find $||u||_1, ||u||_2, ||u||_3, ||u||_\infty$.

   $||u||_2 = \sqrt{0^2 + (-4)^2 + (3)^2} = \sqrt{25} = 5$. $||u||_3 = \sqrt[3]{|0|^3 + |-4|^3 + |3|^3} = \sqrt[3]{91}$. $||u||_\infty = \max\{|0|, |-4|, |3|\} = 4$. $||u||_1 = |0| + |-4| + |3| = 7$.

The remaining questions concern vector space $V = P_2(x)[0, 1]$, the set of polynomials of degree at most 2 on interval $[0, 1]$, with inner product given by $<u, v>= \int_0^1 u(x)v(x)dx$. Set $f(x) = 2, g(x) = 3x$.

3. Find the angle between $f(x), g(x)$.

   $||f||^2 =< f(x), f(x) >= \int_0^1 4dx = 4x|_0^1 = 4 - 0 = 4$, so $||f|| = 2$.
   $||g||^2 =< g(x), g(x) >= \int_0^1 9x^2dx = 3x^3|_0^1 = 3 - 0 = 3$, so $||g|| = \sqrt{3}$.
   $< f(x), g(x) >= \int_0^1 6xdx = 3x^2|_0^1 = 3 - 0 = 3$.
   Hence $\cos\theta = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$, so $\theta = \pi/6 = 30°$.

4. Use the Gram-Schmidt process to find an orthogonal basis for the space $Span(\{f(x), g(x)\})$.

   We start with $u_1 = f(x)$. Then (reusing the calculations from problem 3), $u_2 = g(x) - proj(g(x), u_1(x)) = g(x) - \frac{<g(x),u_1(x)>}{<u_1(x),u_1(x)>}u_1(x) = g(x) - \frac{<g(x),f(x)>}{<f(x),f(x)>}f(x) = 3x - \frac{3}{4}2 = 3x - 3/2$. Hence the desired orthogonal basis is $\{2, 3x - 1.5\}$.

5. Find a basis for $Span(\{f(x)\})^\perp$.

   Since V is 3-dimensional, and $Span(\{f(x)\})$ is 1-dimensional, $Span(\{f(x)\})^\perp$ is 2-dimensional. Hence we need to find any two linearly independent vectors in $Span(\{f(x)\})^\perp$; that is, we need to find any two linearly independent vectors, each orthogonal to $f(x)$.

   Method 1: We already have one, from problem 4, namely $3x - 1.5$. We need to find another. Let's start anywhere, say with $h(x) = x^2$, and calculate $< h(x), f(x) >= \int_0^1 2x^2dx = 2/3x^3|_0^1 = 2/3 - 0 = 2/3$. Hence we have $h(x) - proj(h(x), f(x)) = h(x) - \frac{<h(x),f(x)>}{<f(x),f(x)>}f(x) = x^2 - \frac{2/3}{4}2 = x^2 - 1/3$. This is linearly independent with $3x - 1.5$, hence $\{3x - 3/2, x^2 - 1/3\}$ is a desired basis.

   Method 2: Let's find the general form of a vector orthogonal to f. Consider $h(x) = a + bx + cx^2$, and take $< h(x), f(x) >= \int_0^1 2a+2bx+2cx^2dx = 2ax+bx^2+(2/3)cx^3|_0^1 = 2a+b+(2/3)c$. For h, f to be orthogonal, we must have $2a+b+(2/3)c = 0$. We may now make any choices for a, b, c to satisfy this linear equation, e.g. $a = 1, b = -2$ (giving $h(x) = 1 - 2x$), $c = 3, b = -2$ (giving $h(x) = -2x + 3x^2$). So $\{1 - 2x, -2x + 3x^2\}$ is a basis.