Math 254 Exam 7 Solutions

1. Carefully state the definition of “linearly dependent”. Give two examples, each from $M_{2,2}$.

A set of vectors is linearly dependent if there is some nondegenerate linear combination yielding the zero vector. Many examples are possible, such as $\{(0 \ 0 \ 0), \{1 \ 1\}, \{(2 \ 2\} \}, \{(1 \ 1), \{(1 \ 1\}, \{(0 \ -1\} \}.$ (note that $(1 \ 1) - (1 \ 0) + (0 \ -1) = (0 \ 0)$.)

2. Let $u = (0, -4, 3)$, a vector in $\mathbb{R}^3$. Find $\|u\|_1, \|u\|_2, \|u\|_3, \|u\|_\infty$.

\[
\|u\|_2 = \sqrt{0^2 + (-4)^2 + 3^2} = \sqrt{25} = 5. \quad \|u\|_3 = \sqrt[3]{|0|^3 + |4|^3 + |3|^3} = \sqrt[3]{91}. \quad \|u\|_\infty = \max\{|0|, |-4|, |3|\} = 4. \quad \|u\|_1 = |0| + |-4| + |3| = 7.
\]

The remaining questions concern vector space $V = P_2(x)[0, 1]$, the set of polynomials of degree at most 2 on interval $[0, 1]$, with inner product given by $\langle u, v \rangle = \int_0^1 u(x)v(x)dx$. Set $f(x) = 2, g(x) = 3x$.

3. Find the angle between $f(x), g(x)$.

\[
\begin{align*}
\|f\|^2 &= \langle f(x), f(x) \rangle = \int_0^1 4dx = 4x^4 \bigg|_0^1 = 4 = 4, \quad \text{so } \|f\| = 2. \\
\|g\|^2 &= \langle g(x), g(x) \rangle = \int_0^1 9x^2dx = 3x^3 \bigg|_0^1 = 3 = 3, \quad \text{so } \|g\| = \sqrt{3}. \\
\langle f(x), g(x) \rangle &= \int_0^1 6xdx = 3x^2 \bigg|_0^1 = 3 = 3 = 3. \\
\text{Hence } \cos \theta &= \frac{\|f\| \cdot \|g\|}{\|f\| \cdot \|g\|} = \frac{\sqrt{3}}{2}, \quad \text{so } \theta = \pi/6 = 30^\circ.
\end{align*}
\]

4. Use the Gram-Schmidt process to find an orthogonal basis for the space $\text{Span} \{\{f(x), g(x)\}\}$.

We start with $u_1 = f(x)$. Then (reusing the calculations from problem 3), $u_2 = g(x) - \text{proj}_{u_1}(g(x), u_1(x)) = g(x) - \frac{\langle g(x), u_1(x) \rangle}{\|u_1(x)\|^2}u_1(x) = g(x) - \frac{\langle g(x), f(x) \rangle}{\langle f(x), f(x) \rangle}f(x) = 3x - \frac{3}{4} = 3x/2$. Hence the desired orthogonal basis is $\{2, 3x - 1.5\}$.

5. Find a basis for $\text{Span} \{\{f(x)\}\}^\perp$.

Since $V$ is 3-dimensional, and $\text{Span} \{\{f(x)\}\}$ is 1-dimensional, $\text{Span} \{\{f(x)\}\}^\perp$ is 2-dimensional. Hence we need to find any two linearly independent vectors in $\text{Span} \{\{f(x)\}\}^\perp$; that is, we need to find any two linearly independent vectors, each orthogonal to $f(x)$.

Method 1: We already have one, from problem 4, namely $3x - 1.5$. We need to find another. Let’s start anywhere, say with $h(x) = x^2$, and calculate $\langle h(x), f(x) \rangle = \int_0^1 2x^2dx = 2/3x^3 \bigg|_0^1 = 2/3 - 0 = 2/3$. Hence we have $h(x) - \text{proj}_{f(x)}(h(x), f(x)) = h(x) - \frac{\langle h(x), f(x) \rangle}{\langle f(x), f(x) \rangle}f(x) = x^2 - \frac{2/3}{2/3} = x^2 - 1/3$. This is linearly independent with $3x - 1.5$, hence $\{3x - 3/2, x^2 - 1/3\}$ is a desired basis.

Method 2: Let’s find the general form of a vector orthogonal to $f$. Consider $h(x) = a + bx + cx^2$, and take $\langle h(x), f(x) \rangle = \int_0^1 2a + 2bx + 2cx^2dx = 2ax + bx^2 + (2/3)cx^3 \bigg|_0^1 = 2a + b + (2/3)c$. For $h, f$ to be orthogonal, we must have $2a + b + (2/3)c = 0$. We may now make any choices for $a, b, c$ to satisfy this linear equation, e.g. $a = 1, b = -2$ (giving $h(x) = 1 - 2x$), $c = 3, b = -2$ (giving $h(x) = -2x + 3x^2$). So $\{1 - 2x, -2x + 3x^2\}$ is a basis.