Math 254 Exam 4 Solutions

1. Carefully state the definition of “subspace”. Give two examples, each from \( \mathbb{R}^2 \).

   A subspace is a subset of a vector space that is, itself, a vector space. Many examples are possible, such as \( \{(0, 0)\} \) (zero-dimensional), Span\( \langle S \rangle \) for \( S = \{(1, 2)\} \) (one-dimensional), or \( \mathbb{R}^2 \) itself (two-dimensional).

2. Carefully state any five of the eight vector space axioms.

   These are listed on p.152 of the text. It is not important how you number them; however it is important that you give the English text correctly. “For any vectors \( u, v, w \) in \( V \), \( (u + v) + w = u + (v + w) \)” is correct, but the equation “\( (u + v) + w = u + (v + w) \)” alone is incorrect.

3. Let \( S = \{ f(x) : f(3) = 1 \} \subseteq \mathbb{R}[x] \) be the set of all polynomials \( f(x) \) satisfying \( f(3) = 1 \). Determine, with justification, whether this is a vector space.

   Since \( S \) is a subset of a vector space, to be a subspace \( S \) must satisfy three properties. It must contain the zero vector, it must be closed under vector addition, and closed under scalar multiplication. \( S \) satisfies none of these three properties, and it’s enough to pick your favorite to disprove. Just for fun, I will disprove all three: (1) \( f(x) = 0 \) does not satisfy \( f(3) = 1 \), so \( 0 \) is not in \( S \); (2) \( f(x) = 1 \) and \( g(x) = x/3 \) are both in \( S \), but \( (f + g)(x) = 1 + x/3 \) is not in \( S \) since \( (f + g)(3) = 2 \); (3) \( f(x) = 1 \) is in \( S \) but \( 5f(x) = 5 \) is not in \( S \).

4. Determine, with justification, whether \( (1, 2) \) is in the rowspace of \( M = \begin{bmatrix} 2 & 3 \\ 6 & 9 \end{bmatrix} \).

   The rowspace of \( M \) is also the rowspace of \( \begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix} \), obtained via \( R_2 = R_2 - 3R_1 \), which is Span\( \langle (2, 3) \rangle = \{ t(2, 3) : t \in \mathbb{R} \} \). If \( (1, 2) \) were in this subspace, then for some \( t \) we would have \( (1, 2) = (2t, 3t) \), and hence \( 2t = 1 \) and \( 3t = 2 \). This is impossible, so the answer is “no”.

5. Set \( V = \mathbb{R}^3 \). Give any two subspaces \( U_1, U_2 \) such that \( U_1 \oplus U_2 = V \).

   Two type of solutions are possible. The “trivial” solution is \( U_1 = \{(0, 0, 0)\}, U_2 = \mathbb{R}^3 \) (or the other way around). Otherwise, one of \( U_1, U_2 \) will be one-dimensional and the other will be two-dimensional. Many examples are possible, for example \( U_1 = \text{Span}(\{(1, 0, 0)\}) = \{(a, 0, 0) : a \in \mathbb{R} \}, U_2 = \text{Span}(\{(0, 1, 0), (0, 0, 1)\}) = \{(0, b, c) : b, c \in \mathbb{R} \} \). What is important is that \( U_1, U_2 \) are both subspaces of \( \mathbb{R}^3 \), that \( U_1 + U_2 = \mathbb{R}^3 \), and that \( U_1 \cap U_2 = \{0\} \).