Math 254 Exam 3 Solutions

1. Carefully state the definition of "dependent". Give two examples, each from $\mathbb{R}^5$.

   A set of vectors is dependent if a nondegenerate linear combination yields the zero vector. Many examples are possible, such as $\{(0,0,0,0,0)\}$ or $\{((0,0,0,1,0),(0,0,0,2,0)\}$.

2. Suppose that A is a square matrix. Prove that

   (a) if A is invertible, then A^T is invertible; and
   (b) if A^T is invertible, then A is invertible.

   (a) If A is invertible, then there is some matrix B with $AB = I$. Taking transposes, we get $(AB)^T = I^T = I$, but $B^T A^T = (AB)^T$. Hence A^T is invertible, with inverse B^T.
   (b) If A^T is invertible, we apply part (a) to conclude that $(A^T)^T$ is invertible, but that is just A again.
   ALTERNATE, BORING, SOLUTION for (b): Just repeat (a). If A^T is invertible, then there is some matrix B with $A^T B = I$. Taking transposes, we get $(A^T B)^T = I^T = I$, but $(A^T B)^T = B^T (A^T)^T = B^T A$. Hence A is invertible, with inverse B^T.
   ALTERNATE SOLUTION for (a): If A is invertible, then by Thm 3.7 in the text, $A = E_1 E_2 \cdots E_k$, a product of elementary row matrices. But then $A^T = (E_1 E_2 \cdots E_k)^T = E_k^T \cdots E_2^T E_1^T$, which is again a product of elementary row matrices and is thus invertible by Thm 3.7 again.

3. Let $A = \begin{bmatrix} 1 & 6 & 1 \\ 4 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Write $A = B + C$, where B is symmetric and C is skew-symmetric.

   $A^T = \begin{bmatrix} 1 & 4 & 1 \\ 6 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, so $B = (1/2)(A + A^T) = \begin{bmatrix} 1 & 5 & 1 \\ 5 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $C = (1/2)(A - A^T) = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

4. Let $A = \begin{bmatrix} 1 & 6 & 1 \\ 4 & 2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$. Find A^{-1}, if it exists.

   $\begin{bmatrix} 1 & 6 & 1 & 1 & 0 & 0 \\ 4 & 2 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 4 & 2 & 0 & 0 & 1 & 0 \\ 1 & 6 & 1 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & -4 \\ 0 & 6 & 1 & 1 & 0 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & -4 \\ 0 & 0 & 1 & 1 & -3 & 11 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1/2 & -2 \\ 0 & 0 & 1 & 1 & -3 & 11 \end{bmatrix}$.
   $\{R_1 \leftrightarrow R_3\}$, $\{-4R_1 + R_2 \to R_2, -R_1 + R_3 \to R_3\}$, $\{-3R_3 + R_3 \to R_3\}$, $\{(1/2)R_2 \to R_2\}$.
   Hence $A^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1/2 & -2 \\ 1 & -3 & 11 \end{bmatrix}$.

5. Let $B = \begin{bmatrix} -2 & -1 \\ 4 & 3 \end{bmatrix}$. Find the LU factorization of B.

   To make B upper triangular, we multiply by elementary matrix $E = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$, i.e. $EB = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix} = U$. Hence $E^{-1}EB = E^{-1}U$, so we take $L = E^{-1} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$, and $B = LU$.
   ALTERNATE SOLUTION: Using the ideas from section 3.9 in the book, we have just one multiplier $m_{21} = 2$, after which $U = \begin{bmatrix} -2 & -1 \\ 0 & 1 \end{bmatrix}$ and $L = \begin{bmatrix} 1 & 0 \\ -m_{21} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}$.