Math 254 Exam 10 Solutions

1. Carefully state the definition of "dimension". Give two subspaces from $\mathbb{R}^3$, a zero-dimensional one and a one-dimensional one.

   The dimension of a vector space is the number of vectors in any basis. The only zero-dimensional subspace of $\mathbb{R}^3$ is $\{(0,0,0)\}$. The one-dimensional subspaces of $\mathbb{R}^3$ are all $Span(v)$, for any nonzero vector v, such as $v = (1,2,3)$.

2. Let A be a square matrix whose first row consists entirely of zeroes. Prove that every such matrix A has zero determinant.

   Solution 1: Using the Laplace expansion on the first row, $|A| = a_{11}A_{11} + a_{12}A_{12} + \cdots + a_{1n}A_{1n} = 0A_{11} + 0A_{12} + \cdots + 0A_{1n} = 0$.

   Solution 2: Using the general formula, $|A| = \sum_\sigma (\text{sgn } \sigma) a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{n\sigma(n)}$. But $a_{1\sigma(1)} = 0$ regardless of σ, so each summand is zero and hence the sum is zero.

   Solution 3: Consider A as a block (lower) triangular matrix, with a 1×1 block and an $(n-1) \times (n-1)$ block. $|A|$ is the product of the determinants of the two blocks; since one is zero then $|A| = 0$.

   Solution 4: Put A into row echelon form. The result is is upper triangular, so we can find the determinant by multiplying along the diagonal. But the bottom right corner is zero, since there is at least one all-zero row in the row echelon form. After correcting for our row echelon operations, we will still have $|A| = 0$.

The remaining problems all concern the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 4 \\ 4 & 1 & 5 \end{bmatrix}$.

3. Calculate $|A|$ using the formula for 3×3 determinants.

   We write $\begin{bmatrix} 1 & 2 & 3 & 1 & 2 \\ -2 & 0 & 4 & -2 & 0 \\ 4 & 1 & 5 & 4 & 1 \end{bmatrix}$, so $|A| = (1)(0)(5) + (2)(4)(4) + (3)(-2)(1) - [(3)(0)(4) + (1)(4)(1) + (2)(-2)(5)] = 0 + 32 - 6 - [0 + 4 - 20] = 32 - 6 + 16 = 42$.

4. Calculate $|A|$ by expanding on the second column.

   $|A| = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32} = 2A_{12} + 0A_{22} + 1A_{32} = 2(-1)\begin{vmatrix} -2 & 4 \\ 4 & 5 \end{vmatrix} + 1(-1)\begin{vmatrix} 1 & 3 \\ -2 & 4 \end{vmatrix} = -2(-10-16) - (4+6) = 52 - 10 = 42$.

5. Calculate $|A|$ by making A triangular with elementary row operations.

   We pivot on the $(1,1)$ entry, via $R2 = R2 + 2R1, R3 = R3 - 4R1$, to get $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & -7 & -7 \end{bmatrix}$. Neither of these changes the determinant. Next, we do $R3 = R3 + (7/4)R2$, which also does not change the determinant. This gives $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 10 \\ 0 & 0 & \frac{42}{4} \end{bmatrix}$. Hence $|A| = (1)(4)(\frac{42}{4}) = 42$.