1. Carefully state the definition of “basis”. Give two examples from \( \mathbb{R}^2 \).

A basis is a set of vectors that is both independent and spanning. Equivalently, a basis is a maximal set of independent vectors. Equivalently, a basis is a minimal set of spanning vectors. Many examples are possible, such as \( \{(1,0), (0,1)\} \) or \( \{(1,1), (2,3)\} \). All must contain exactly two, linearly independent, vectors.

Problems 2 and 3 both concern the matrix \( A = \begin{pmatrix} 2 & -4 & 6 & 0 & 4 \\ 1 & -2 & 3 & 0 & 2 \\ -1 & 2 & -3 & 1 & -1 \\ -2 & 4 & -6 & 2 & -2 \\ 3 & -6 & 9 & -3 & 3 \end{pmatrix} \).

2. Set \( S = \text{Rowspan}(A) \). Find a basis for \( S \), and determine its dimension.

The first two rows are two pivots, hence \( S \) is two dimensional. Many bases are possible; the natural one is the nonzero rows of the echelon matrix: \( \{(1,−2,3,0,2), (0,0,0,1,1)\} \). However any two independent elements of the rowspan would also work, such as two independent rows of \( A \) itself: \( \{(2,−4,6,0,4), (−1,2,−3,1,−1)\} \). The first two rows of \( A \) will NOT work, since they are dependent.

3. Set \( T = \text{Columnspan}(A) \). Find a basis for \( T \), and determine its dimension.

The rowspace and columnspan have the same dimension, hence \( T \) is two dimensional. The pivots of the row echelon form of \( A \) are in the first and fourth columns, hence the first and fourth columns of \( A \) form a basis for the columnspan: \( \{(2,1,−1,−2,3)^T, (0,0,1,2,−3)^T\} \). This is not the only basis; any two independent elements of the columnspan would also work.

Problems 4 and 5 both concern the vector spaces \( A = \text{Span}(\{(2,0,1), (1,−1,3)\}) \) and \( B = \text{Span}(\{(5,1,0), (0,4,−10)\}) \). Both are subspaces of \( \mathbb{R}^3 \).

4. Find a basis for \( A + B \), and determine its dimension.

We begin by putting the generating vectors in a matrix, then putting this matrix into echelon form: \( \begin{pmatrix} 2 & 1 & 5 & 0 \\ 0 & -1 & 1 & 4 \\ 1 & 3 & 0 & -10 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & -5 & 5 & 20 \\ 0 & -1 & 1 & 4 \\ 1 & 3 & 0 & -10 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 3 & 0 & -10 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix} \). This has two pivots, hence \( A + B \) is two dimensional. The pivots are in the first two columns, hence a basis is \( \{(2,0,1), (1,−1,3)\} \) (other bases are possible). Note: this solution has the matrix as \( 3 \times 4 \), putting the vectors into columns. It is equally correct to put the vectors into rows, giving a \( 4 \times 3 \) matrix.

5. Find a basis for \( A \cap B \), and determine its dimension.

\[
\text{dim}(A + B) + \text{dim}(A \cap B) = \text{dim}(A) + \text{dim}(B).
\]
We have \( \text{dim}(A) = \text{dim}(B) = \text{dim}(A + B) = 2 \), hence we can solve and determine \( \text{dim}(A \cap B) = 2 \). Hence \( A \cap B = A = B = A + B \), so a basis for \( A \cap B \) is any basis for \( A \) (or \( B \)), such as \( \{(2,0,1), (1,−1,3)\} \).