Math 254-2 Exam 4 Solutions

1. Carefully state the definition of “subspace”. Give two examples from \( \mathbb{R}^2 \).

   A subspace is a vector space, that is contained within another vector space. Many examples are possible from \( \mathbb{R}^2 \): \( \{0\} \), \( \mathbb{R}^2 \) itself, \( \text{Span}(v) \) for any vector \( v \) (in \( \mathbb{R}^2 \)), the solution set to any \( 2 \times 2 \) homogeneous system of linear equations. Note that \( \mathbb{R}^1 \) is NOT a subspace, since none of its vectors are in \( \mathbb{R}^2 \).

2. Carefully state five of the eight vector space axioms.

   It is important not only to have the axioms right, but the quantifiers (for all vectors \( $u$, \( v $, etc.) You may find a list of the axioms on p.152 of the text. The names the book gives them (e.g. \( A_3 \)) are unimportant.

3. Let \( S = \{ f(x) : f(17) = 0 \} \subseteq \mathbb{R}[x] \) be the set of all polynomials that are zero at \( x = 17 \). Prove that this is a vector space.

   \( S \) is a nonempty subset of \( \mathbb{R}[x] \), so by Thm. 4.2 we need only check closure. If \( f, g \) are both in \( S \), then \( f(17) = g(17) = 0 \). \( (f + g)(17) = f(17) + g(17) = 0 + 0 = 0 \), so \( f + g \) is in \( S \).

   Alternate proof: Instead of two steps, closure can be verified in one step. If \( f, g \) are both in \( S \), and \( a \) is any scalar, then \( (cf)(17) = cf(17) = c0 = 0 \), so \( cf \) is in \( S \). \( S \) satisfies closure, hence is a subspace.

4. Determine, with justification, whether \((1, 1, 1)\) is in \( \text{Span}(S) \), for \( S = \{(1, 2, 1), (0, 3, 2), (2, 1, 0)\} \).

   The answer is yes, precisely when there are solutions to the linear system

   \[
   \begin{bmatrix}
   1 & 2 & 1 \\
   2 & 3 & 1 \\
   1 & 2 & 0
   \end{bmatrix}
   \begin{bmatrix}
   a \\
   b \\
   c
   \end{bmatrix} =
   \begin{bmatrix}
   1 \\
   1 \\
   1
   \end{bmatrix}
   \]

   We solve this in the usual way, with the augmented matrix

   \[
   \begin{bmatrix}
   1 & 2 & 1 & 1 \\
   2 & 3 & 1 & 1 \\
   1 & 2 & 0 & 1
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 2 & 1 & 1 \\
   0 & 1 & -1 & 1/3 \\
   0 & 2 & -2 & 0
   \end{bmatrix}
   \rightarrow
   \begin{bmatrix}
   1 & 0 & 2 & 1 \\
   0 & 1 & -1 & 1/3 \\
   0 & 0 & 0 & 2/3
   \end{bmatrix}
   \]

   The last equation is \( 0 = 2/3 \), which has no solutions. Hence \((1, 1, 1)\) is NOT in \( \text{Span}(S) \).

   There is no linear combination of the elements of \( S \), that yields \((1, 1, 1)\).

5. Let \( W_1 = \text{Span}(S) \), for \( S = \{(1 \, 1 \, 1), (0 \, 0 \, 0)\} \). Let \( W_2 = \text{Span}(T) \), for \( T = \{(0 \, 0 \, 0), (0 \, 0 \, 0)\} \). Prove that \( W_1 \oplus W_2 = M_{2 \times 2}(\mathbb{R}) \) (the set of all \( 2 \times 2 \) matrices).

   Note that \( W_1 = \{ w(1 \, 0 \, 0 \, 1) + x(0 \, 0 \, 0 \, 1) \} \) for \( \{ w(1 \, 0 \, 0 \, 1) \}, W_2 = \{ y(0 \, 0 \, 1 \, 0) + z(0 \, 0 \, 0 \, 0) \} = \{ (0 \, 0 \, 0 \, 0) \} \).

   Solution 1: We need to express every \( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \) uniquely as a sum of some vector from \( W_1 \) and some vector from \( W_2 \). We have \( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = w(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}) + x(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}) \) for \( w(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}), x(\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}) \). The equations we get are \( w + y = a, w = b, x + z = c, x = d \). These equations have a unique solution, namely \( w = b, y = a - b, x = d, z = c - d \); hence \( \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} b \\ b \\ c-d \\ 0 \end{bmatrix} + \begin{bmatrix} a-b \\ 0 \\ 0 \\ 0 \end{bmatrix} \), where the first matrix is in \( W_1 \) and the second is in \( W_2 \).

   Solution 2: We use Thm 4.11, which requires two things: (1) \( M_{2 \times 2} = W_1 + W_2 \), and (2) \( W_1 \cap W_2 = \{0\} \). To prove (1), we use the calculation from the first solution, although it is no longer important to have a unique decomposition. To prove (2), we need to find all matrices common to both \( W_1 \) and \( W_2 \). \( \begin{bmatrix} w \\ w \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \). The only solution is \( w = x = y = z = 0 \), hence \( W_1 \cap W_2 = \{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \} = \{0\} \).