1. Carefully define the term “basis”. Give two examples in $\mathbb{R}^2$.

A basis is a set of vectors that is both independent and spanning. Equiva-
ently, it is a set of vectors that is maximal and independent. Equivalently,
it is a set of vectors that is minimal and spanning. Examples in $\mathbb{R}^2$ include
$\{(1, 0), (0, 1)\}$ and $\{(1, 0), (1, 1)\}$.

For the remaining problems, consider the matrix $A = \begin{pmatrix} 2 & -1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{pmatrix}$.

Hint: all solutions can be expressed with integers.

2. Calculate the characteristic polynomial $\Delta(t)$ (or $p(\lambda)$) for $A$.

$$p(\lambda) = |\lambda I - A| = \begin{vmatrix} \lambda - 2 & 1 & -1 \\ 0 & \lambda - 1 & -1 \\ -1 & 0 & \lambda - 2 \end{vmatrix} = (\lambda - 1)((\lambda - 2)^2 - 1) = (\lambda - 1)(\lambda - 3)(\lambda - 1) = \lambda^3 - 5\lambda^2 + 7\lambda - 3.$$ 

3. Find all the eigenvalues of $A$.

We solve $0 = p(\lambda) = (\lambda - 3)(\lambda - 1)^2$. It has two solutions: $\lambda = 3$ and $\lambda = 1$ (a double root). Hence $1, 3$ are the eigenvalues of $A$.

4. For each eigenvalue, find a maximal independent set of eigenvectors.

For $\lambda = 1$, $A - \lambda I = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This has row canonical form $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since there are two pivots, the rank is two and the nullity is one; hence the eigenspace is one-dimensional. If $(x, y, z)^T$ is in the nullspace, then $x + z = 0, y = 0$. Hence a basis for the eigenspace is $(1, 0, -1)^T$.

For $\lambda = 3$, $A - \lambda I = \begin{pmatrix} -1 & -1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. This has row canonical form $\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since there are two pivots, the rank is two and the nullity is one; hence the eigenspace is one-dimensional. If $(x, y, z)^T$ is in the nullspace, then $x - z = 0, y = 0$. Hence a basis for the eigenspace is $(1, 0, 1)^T$.

5. For each eigenvalue, give its algebraic and geometric multiplicity. Is $A$ diagonalizable?

What is the Jordan form of $A$?

$\lambda = 3$ is a single root, hence its algebraic multiplicity (and thus geometric multiplicity) is 1. $\lambda = 1$ is a double root, hence its algebraic multiplicity is 2. However, its geometric multiplicity is 1, as calculated in the previous problem. Hence $A$ is not diagonalizable, and it has Jordan form $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ or $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$.