1. Carefully state the definition of “spanning”. Give two examples for \( \mathbb{R}^2 \).

A set of vectors \( S \) is spanning if every vector in the vector space can be achieved through linear combinations of \( S \). Equivalently, \( S \) is spanning if \( \text{span}(S) \) is the whole vector space. Many examples are possible. Any basis, such as \( \{(1,0),(0,1)\} \), will work. But other examples are possible too, such as \( \{(1,1),(1,2),(1,3)\} \).

2. Let \( u = [1 \ 2 \ 3] \), and \( v = [0 \ 7 \ 15] \). For each of the following, determine what type they are (undefined, scalar, matrix/vector). If a matrix/vector, specify the dimensions.

**DO NOT CALCULATE ANY NUMBERS.**

(a) \( uu \)  (b) \( uv^T \)  (c) \( u^Tvu^T \)  (d) \( u \times v \)  (e) \( (u \times v) \cdot u \)

\( u, v \) are \( 1 \times 3 \); \( u^T, v^T \) are \( 3 \times 1 \). Hence \( uu \) has pattern \( (1 \times 3)(1 \times 3)(1 \times 3) \); neither matrix multiplication is possible, hence (a) is undefined. \( uv^T \) has \( (1 \times 3)(3 \times 1)(1 \times 3) \); both matrix multiplications are possible, and the result of (b) is a \( 1 \times 3 \) matrix (or a row 3-vector). \( u^Tvu^T \) has \( (3 \times 1)(1 \times 3)(3 \times 1) \); both matrix multiplications are possible, and the result of (c) is a \( 3 \times 1 \) matrix (or a column 3-vector). \( u \times v \) gives a scalar, hence (d) is undefined since cross product requires two vectors. (e) is a scalar, because \( (u \times v) \) is a 3-vector, hence its dot product with \( u \) can be calculated and is a scalar.

3. Let \( u = (1,2,3) \), and \( v = (15,-7,0) \). Are these vectors orthogonal?

Be sure to justify your answer.

We calculate \( u \cdot v = 1(15) + 2(-7) + 3(0) = 1 \). Since this is nonzero, these vectors are not orthogonal.

4. For \( A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 3 \end{bmatrix} \) and \( B = \begin{bmatrix} 2 & -1 \\ 0 & 1 \\ 1 & 5 \end{bmatrix} \), calculate \( AB \) and \( BA \).

\[ AB = \begin{bmatrix} 0+0 & 1-0 & -1+0 \\ -1+0 & 0-0 & 3-0 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \].  \( BA = \begin{bmatrix} 0-1 & 2+0 & -2+3 \\ 0+1 & 0+0 & 0-3 \\ 0+5 & 1+0 & -1+15 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 1 \\ 1 & 0 & -3 \\ 5 & 1 & 14 \end{bmatrix} \).

5. For \( u = (1,0,2) \) and \( v = (0,-3,1) \), calculate \( u \times v \) and \( v \times u \).

Method 1, determinant formula: \( u \times v = \begin{vmatrix} i & j & k \\ 0 & 0 & 1 \\ -1 & 0 & 3 \end{vmatrix} = (0+6)i - (1-0)j + (0-0)k = 6i - 1j - 3k = (6,-1,-3) \)

\( v \times u = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ -1 & 0 & 3 \end{vmatrix} = (-6+0)i - (0-1)j + (0+3)k = -6i + 1j + 3k = (-6,1,3) \)

Method 2, i,j,k technique: \( u \times v = (i+2k) \times (-3j+1) = -3(i \times j) + (i \times k) - 6(k \times j) + 2(k \times k) = -3(i \times j) + (i \times k) - 6(i \times j) + 2(k \times k) = -6(i \times j) + 3(k \times k) = -6(-1) + 3(6) = 6 + 18 = 24 \)

\( v \times u = (-3j + k) \times (i + 2k) = -3(j \times i) - 6(j \times k) + (k \times i) + 2(k \times k) = -6(k \times v) = -6(3) = -18 \)

\( u \times v = (i \times k) - (j \times k) = 2k - 6i + j = (-6,1,3) \)