Math 254 Exam 9 Solutions

1. Carefully define the term "basis". Give two examples in $\mathbb{R}^2$.

   A basis is a set of vectors that is independent and maximal. Four examples are $\{(1,0),(0,1)\}$, $\{(1,1),(3,2)\}$, $\{(1,0),(3,7)\}$, $\{(-8,8),(-8,7)\}$.

2. Consider the basis $S = \{u_1, u_2\} = \{(1,-2),(2,-5)\}$ of $\mathbb{R}^2$, and the linear operator $F(x,y) = (2x+3y, 4x-5y)$. Find the matrix representation $[F]_S$.

   The change-of-basis matrix from the standard basis E to S is $P = \left[\begin{smallmatrix} 1 & 2 \\ -2 & -5 \end{smallmatrix}\right]$. We also have $[F]_E = \left[\begin{smallmatrix} 2 & 3 \\ 4 & -5 \end{smallmatrix}\right]$. We now calculate $P^{-1} = \left[\begin{smallmatrix} 5 & 2 \\ -2 & -1 \end{smallmatrix}\right]$, and now $[F]_S = P^{-1}[F]_E P = \left[\begin{smallmatrix} 5 & 2 \\ -2 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 2 & 3 \\ 4 & -5 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 & 2 \\ -2 & -5 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 8 & 11 \\ -6 & -11 \end{smallmatrix}\right]$.

3. Let V be the vector space of functions that have as a basis $S = \{u_1, u_2, u_3, u_4, u_5\} = \{1, \sin\theta, \cos\theta, \sin 2\theta, \cos 2\theta\}$. Let D be the differential operator on V. Find the matrix representation $[D]_S$.
   BONUS: What is the nullity and rank of D?

   We calculate $D(u_1) = 0, D(u_2) = u_3, D(u_3) = -u_2, D(u_4) = 2u_5, D(u_5) = -2u_4$. Therefore $[D]_S = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 2 & 0 \end{bmatrix}$. BONUS: $\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ is $[D]_S$ in row canonical form. This is rank four; D must have the same rank. Since S has dimension 5, the nullity of D is $5 - 4 = 1$.

4. Set $A = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 2 \end{smallmatrix}\right]$. Find two other matrices similar to A.

   Many solutions exist; choose any invertible matrix P, calculate P^{-1}, and $P^{-1}AP$ will be similar to A. For $P = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 2 \end{smallmatrix}\right]$, we have $P^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1/2 \end{smallmatrix}\right]$, and $P^{-1}AP = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 2 \end{smallmatrix}\right] \sim A$. For $P = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$, $P^{-1} = \left[\begin{smallmatrix} -2 & 1 \\ 3/2 & -1/2 \end{smallmatrix}\right]$, and $P^{-1}AP = \left[\begin{smallmatrix} -2 & -4 \\ 3 & 5 \end{smallmatrix}\right] \sim A$.

5. Prove that, for any square matrices A, B, if A is similar to B, then B must be similar to A.

   1. Suppose that A is similar to B.
   2. Hence, there is some invertible matrix, P, such that $A = P^{-1}BP$.
   3. Multiply on the left by P, and on the right by P^{-1} to get $PAP^{-1} = PP^{-1}BPP^{-1} = IBI = B$.
   4. Set $Q = P^{-1}$. Q is invertible, and $Q^{-1} = (P^{-1})^{-1} = P$.
   5. There is an invertible matrix, namely Q, such that $B = Q^{-1}AQ$.
   6. Hence, B is similar to A.