1. Carefully define the term “linear mapping”. Give two examples in \( \mathbb{R}^2 \).

A linear mapping is a function \( f : V \rightarrow U \) with \( U, V \) vector spaces, such that \( f(v+w) = f(v) + f(w) \), and \( f(kv) = kf(v) \), for any vectors \( v, w \) and scalar \( k \). Many examples exist; e.g. \( f(x, y) = (x, y), f(x, y) = (2x, 0) \), \( f(x, y) = (0, 0) \), \( f(x, y) = (-2x + 3y, 4x - 2y) \).

2. Consider the mapping \( f : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) defined by \( f(x, y, z) = (x + 2y, 0, -3y) \). Is this linear?

First, \( f(x + x', y + y', z + z') = ((x + x') + 2(y + y'), 0, -3(y + y')) = (x + 2y, 0, -3y) + (x' + 2y', 0, -3y') = f(x, y, z) + f(x', y', z') \). Second, \( f(kx, ky, kz) = (kx + 2ky, 0, -3ky) = k(x + 2y, 0, -3y) = kf(x) \). So, \( f \) satisfies both required properties and is linear.

3. Consider all relations whose domain is \( \{A, B\} \) and whose codomain is \( \{1, 2, 3\} \). For each of the following, either give an example or state that no example exists.

(a) A one-to-one function. Many solutions exist; e.g. \( f(A) = 1, f(B) = 3 \)

(b) An onto function. Impossible; the range will always be at most size 2.

(c) A function that is neither one-to-one nor onto. \( f(A) = f(B) = 1 \). Two other solutions are possible, replacing 1 by another element of the codomain.

(d) A relation whose inverse is a function. Many solutions exist; the original relation may NOT be a function itself. e.g. \( f(A) = 1, f(A) = 3, f(B) = 2 \); this can also be written \( \{(A, 1), (A, 3), (B, 2)\} \).

BONUS: How many relations are there? How many functions are there?

\( \{A, B\} \times \{1, 2, 3\} \) has six elements; therefore, it has \( 2^6 = 64 \) subsets. Each subset is a relation, so there are 64 relations. A function picks one element of the codomain for \( f(A) \), and one for \( f(B) \); there are \( 3 \times 3 = 9 \) functions. If you like questions like these, consider the course MATH 579 Combinatorics.

4. Consider the linear mapping \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) defined by \( f(x, y) = (x - y, x - 2y) \). Find a formula for \( f^{-1} \).

\( f(x, y) = (x - y, x - 2y) = (a, b) \). Hence \( x - y = a, x - 2y = b \); we solve to get \( x = 2a - b, y = a - b \). Hence \( f^{-1}(a, b) = (2a - b, a - b) \); equivalently \( f^{-1}(x, y) = (2x - y, x - y) \).

5. Consider all linear mappings from \( \mathbb{R}^3 \) to \( \mathbb{R}^2 \). What are the possible nullities? What are the possible ranks? Give specific examples illustrating each possible value.

The rank may be 0, 1, or 2 (because the codomain has dimension 2). The dimension of the domain is 3. Therefore, the nullity may be 3, 2, or 1 (because of Theorem 8.6). \( f(x, y, z) = (0, 0) \) has rank 0 and nullity 3. \( f(x, y, z) = (x, 0) \) has rank 1 and nullity 2. \( f(x, y, z) = (x, y) \) has rank 2 and nullity 1. Many other examples are possible.