Math 254 Exam 8 Solutions

1. Carefully define the term "linear mapping". Give two examples in $\mathbb{R}^2$.

   A linear mapping is a function $f : V \to U$ with U, V vector spaces, such that $f(v+w) = f(v)+f(w)$, and $f(kv) = kf(v)$, for any vectors v, w and scalar k. Many examples exist; e.g. $f(x, y) = (x, y), f(x, y) = (2x, 0), f(x, y) = (0, 0), f(x, y) = (-2x + 3y, 4x - 2y)$.

2. Consider the mapping $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $f(x, y, z) = (x + 2y, 0, -3y)$. Is this linear?

   First, $f(x+x', y+y', z+z') = ((x + x') + 2(y + y'), 0, -3(y + y')) = (x+2y, 0, -3y) + (x'+2y', 0, -3y') = f(x, y, z)+f(x', y', z')$. Second, $f(kx, ky, kz) = (kx+2ky, 0, -3ky) = k(x + 2y, 0, -3y) = kf(x)$. So, f satisfies both required properties and is linear.

3. Consider all relations whose domain is $\{A, B\}$ and whose codomain is $\{1, 2, 3\}$. For each of the following, either give an example or state that no example exists.

   (a) A one-to-one function. Many solutions exist; e.g. $f(A) = 1, f(B) = 3$

   (b) An onto function. Impossible; the range will always be at most size 2.

   (c) A function that is neither one-to-one nor onto. $f(A) = f(B) = 1$. Two other solutions are possible, replacing 1 by another element of the codomain.

   (d) A relation whose inverse is a function. Many solutions exist; the original relation may NOT be a function itself. e.g. $f(A) = 1, f(A) = 3, f(B) = 2$; this can also be written $\{(A, 1), (A, 3), (B, 2)\}$.

   BONUS: How many relations are there? How many functions are there?

   $\{A, B\} \times \{1, 2, 3\}$ has six elements; therefore, it has $2^6 = 64$ subsets. Each subset is a relation, so there are 64 relations. A function picks one element of the codomain for $f(A)$, and one for $f(B)$; there are $3 \times 3 = 9$ functions. If you like questions like these, consider the course MATH 579 Combinatorics.

4. Consider the linear mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x, y) = (x - y, x - 2y)$. Find a formula for f^{-1}.

   $f(x, y) = (x - y, x - 2y) = (a, b)$. Hence $x - y = a, x - 2y = b$; we solve to get $x = 2a - b, y = a - b$. Hence $f^{-1}(a, b) = (2a - b, a - b)$; equivalently $f^{-1}(x, y) = (2x - y, x - y)$.

5. Consider all linear mappings from $\mathbb{R}^3$ to $\mathbb{R}^2$. What are the possible nullities? What are the possible ranks? Give specific examples illustrating each possible value.

   The rank may be 0,1, or 2 (because the codomain has dimension 2). The dimension of the domain is 3. Therefore, the nullity may be 3, 2, or 1 (because of Theorem 8.6). $f(x, y, z) = (0, 0)$ has rank 0 and nullity 3. $f(x, y, z) = (x, 0)$ has rank 1 and nullity 2. $f(x, y, z) = (x, y)$ has rank 2 and nullity 1. Many other examples are possible.