Math 254 Exam 7b Solutions

1. Carefully define the Linear Algebra term "dependent". Give two examples in $\mathbb{R}^2$.

   A set of vectors is dependent if there is a nondegenerate linear combination yielding $\overline{0}$. Two examples are $\{(0,0)\}$ and $\{(1,2),(2,4),(1,1)\}$.

2. Carefully define the term "orthonormal". Give two examples in $\mathbb{R}^2$.

   A set of vectors is orthonormal if, under an inner product, any pair of different vectors yields zero, but any vector with itself yields one. Using the usual inner product, $\{(1,0)\}$ and $\left\{(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}),(\frac{1}{\sqrt{2}},\frac{-1}{\sqrt{2}})\right\}$ are orthonormal.

3. Let $u=(2,0,-3)$, a vector in $\mathbb{R}^3$. Find $||u||_1, ||u||_2, ||u||_3, ||u||_\infty$.

   $||u||_1 = |2|+|0|+|-3| = 5; ||u||_2 = \sqrt{|2|^2+|0|^2+|-3|^2} = \sqrt{4+9} = \sqrt{13}; ||u||_3 = \sqrt[3]{|2|^3+|0|^3+|-3|^3} = \sqrt[3]{8+27} = \sqrt[3]{35}; ||u||_\infty = \max_i |u_i| = \max\{|2|,|0|,|-3|\} = 3$

4. For the vector space $\mathbb{R}^4$, set $v_1=(1,1,1,1), v_2=(1,1,2,4), v_3=(1,2,-4,-3)$, $S=\text{span}\{v_1,v_2,v_3\}$. Find an orthogonal basis for S.

   Set $w_1=v_1; w_2 = v_2-\text{proj}(v_2,w_1) = (1,1,2,4)-\frac{8}{4}(1,1,1,1) = (-1,-1,0,2)$. Set $w_3 = v_3-\text{proj}(v_3,w_1)-\text{proj}(v_3,w_2) = (1,2,-4,-3)-\frac{-4}{4}(1,1,1,1)-\frac{-9}{6}(-1,-1,0,2) = (\frac{1}{2},\frac{3}{2},-3,1)$. Now $\{w_1,w_2,w_3\}$ is an orthogonal basis for S.

5. For the vector space $\mathbb{R}^3$, set $T=\text{span}\{(1,2,3)\}$. Find an orthogonal basis for $T^\perp$.

   The first step is to find ANY basis for $T^\perp$; that is, two vectors each perpendicular to $(1,2,3)$ that are not linearly dependent themselves. There are many candidates; for example: $(2,-1,0),(3,0,-1),(1,1,-1),(0,3,-2)$. One checks that $(2,-1,0)\cdot(1,2,3) = 0$; similarly, each of the others is perpendicular to $(1,2,3)$. Let's arbitrarily choose $u_1=(2,-1,0), u_2=(3,0,-1)$ to be the vectors we use.

   Now, we perform Gram-Schmidt on these vectors. $w_1=u_1; w_2=u_2-\text{proj}(u_2,w_1) = (3,0,-1)-\frac{6}{5}(2,-1,0) = (\frac{3}{5},\frac{6}{5},-1)$. Now $\{w_1,w_2\}$ is the desired orthogonal basis for $T^\perp$. If desired, check that each of w_1,w_2 is perpendicular to $(1,2,3)$, and that w_1,w_2 are perpendicular to each other.

   Many other answers are possible, depending on which two initial basis vectors are chosen.