Math 254 Exam 7a Solutions

1. Carefully define the term “linear combination”.

   A linear combination of some objects (vectors, variables, etc.) is the sum of those objects, each multiplied by a scalar.

2. Choose ALWAYS or SOMETIMES or NEVER.

   (a) An inner product space is ALWAYS a vector space.
   (b) A normed space is SOMETIMES an inner product space.
   (c) A vector space is SOMETIMES an inner product space.
   (d) An inner product space is ALWAYS a normed space.
   (e) A normed space is ALWAYS a vector space.
   (f) A vector space is SOMETIMES a normed space.

3. Carefully state the three axioms of an inner product.

   (I1) $< a\overline{u_1} + b\overline{u_2}, \overline{v} >= a < \overline{u_1}, \overline{v} > + b < \overline{u_2}, \overline{v} >$
   (I2) $< \overline{u}, \overline{v} >=< \overline{v}, \overline{u} >$
   (I3) $< \overline{u}, \overline{u} > \geq 0$; further, $< \overline{u}, \overline{u} >= 0$ precisely when $\overline{u} = \overline{0}$.

For the next two questions, consider the vector space $P(t)$ with inner product given by $< u, v >= \int_0^1 u(t)v(t)dt$. Let $f(t) = t$, and $g(t) = at + 1$, for some unknown constant a.

4. For which value(s) of a are f and g orthogonal?

   $< f, g >= \int_0^1 at^2 + tdt = at^3/3 + t^2/2\Big|_0^1 = a/3 + 1/2$. f, g are orthogonal precisely when $< f, g >= 0$; this occurs for $a = -3/2$.

5. We want to find which value(s) of a cause f, g to have a $60°$ angle between them. Set up (but do not solve) an equation in a that would answer this question.
   BONUS: Solve the equation.

   $\cos(60°) = 1/2 = \frac{<f,g>}{||f||\ ||g||}$. $< f, g >= a/3 + 1/2$, as calculated before. $||f|| = \sqrt{\int_0^1 t^2 dt} = \sqrt{t^3/3\Big|_0^1} = \sqrt{1/3}$. $||g|| = \sqrt{\int_0^1 (at+1)^2 dt} = \sqrt{\int_0^1 a^2t^2 + 2at + 1dt} = \sqrt{a^2t^3/3 + at^2 + t\Big|_0^1} = \sqrt{a^2/3 + a + 1}$ Hence, we need to solve $1/2 = \frac{a/3+1/2}{\sqrt{1/3}\sqrt{a^2/3+a+1}}$.

   BONUS: $1/2 = \frac{a/3+1/2}{\sqrt{1/3}\sqrt{1/3}\sqrt{a^2+3a+3}} = \frac{a+3/2}{\sqrt{a^2+3a+3}}$. We cross-multiply to get $a + 3/2 = (1/2)\sqrt{a^2+3a+3}$. We now square both sides to get $(1/4)(a^2+3a+3) = (a+3/2)^2 = a^2 + 3a + 9/4$. We rearrange to get $(3/4)a^2 + (9/4)a + (6/4) = 0$. We simplify by multiplying both sides by $(4/3)$ to get $a^2 + 3a + 2 = 0$. We factor this to get $(a+2)(a+1) = 0$ and thus $a = -1, -2$. However, $a = -2$ is extraneous; $(-2)/3 + 1/2 < 0$, so for this value of a, the vectors do not have angle $60°$ (in fact, they have angle $120°$). This leaves only $a = -1$.