Math 254 Exam 6 Solutions

1. Carefully define the Linear Algebra term “independent”.

A set of vectors is independent if no nondegenerate linear combination yields 0.

2. In the vector space $M_{2,3}$ of $2 \times 3$ matrices, set:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 2 & 4 & 7 \\ 10 & 1 & 13 \end{bmatrix}, C = \begin{bmatrix} 1 & 2 & 5 \\ 8 & 2 & 11 \end{bmatrix}$$

Determine whether or not $\{A, B, C\}$ is independent.

Let $E$ be the standard basis for $M_{2,3}$. Then $[A]_E = [1 \, 2 \, 3 \, 4 \, 0 \, 5], [B]_E = [2 \, 4 \, 7 \, 10 \, 1 \, 13], [C]_E = [1 \, 2 \, 5 \, 8 \, 2 \, 11]$. We put these row matrices into a larger matrix, which we then put into echelon form:

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 5 \\ 2 & 4 & 7 & 10 & 1 & 13 \\ 1 & 2 & 5 & 8 & 2 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 0 & 5 \\ 0 & 0 & 1 & 2 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This is seen to have rank 2, hence $\{A, B, C\}$ is dependent.

**ALTERNATE SOLUTION:**

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 3 & 7 & 5 \\ 4 & 10 & 8 \\ 0 & 1 & 2 \\ 5 & 13 & 11 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

This has rank 2, hence $\{A, B, C\}$ is dependent.

3. In the vector space $P_3(x)$ of polynomials of degree at most 3, set $u_1 = x^3 + 3x^2 - 2x + 4$, $u_2 = 2x^3 + 7x^2 - 2x + 5$, $u_3 = x^3 + 5x^2 + 2x - 2$, $u_4 = 2x^3 + 6x^2 - 4x + 5$.

Set $S = \text{span}\{u_1, u_2, u_3, u_4\}$. Find the dimension of $S$, and a basis.

Let $E = \{x^3, x^2, x, 1\}$ be the usual basis for $P_3(x)$. We have $[u_1]_E = [1 \, 3 \, -2 \, 4], [u_2]_E = [2 \, 7 \, -2 \, 5], [u_3]_E = [1 \, 5 \, 2 \, -2], [u_4]_E = [2 \, 6 \, -4 \, 5]$.

We put these row matrices into a larger matrix, which we then put into echelon form:
This has rank 3; hence \( \dim S = 3 \). A basis for \( S \) is \( \{x^3 + 3x^2 - 2x + 4, x^2 + 2x - 3, 1\} \).

**ALTERNATE SOLUTION:**

This has rank 3; hence \( \dim S = 3 \). Because the pivots are in the first, second, and fourth columns, a basis for \( S \) is \( \{u_1, u_2, u_4\} \).

4. In the vector space \( \mathbb{R}^2 \), set \( S = \{(1, 3), (1, 4)\} \), a basis. Find the change-of-basis matrix from \( S \) to the standard basis, and use this matrix to find \( [(5, -3)]_S \).

\[ P = \begin{bmatrix} 1 & 1 \\ 3 & 4 \end{bmatrix} \text{ consists of } S \text{ in column form; } Q = P^{-1} = \begin{bmatrix} 4 & -1 \\ -3 & 1 \end{bmatrix} \text{ is the desired matrix. We find } [(5, -3)]_S = Q \begin{bmatrix} 5 \\ -3 \end{bmatrix} = \begin{bmatrix} 23 \\ -18 \end{bmatrix}. \]

5. In the vector space \( \mathbb{R}^3 \), set \( T = \{(1, 1, 1), (0, 1, 2), (1, 1, 3)\} \), a basis. Find \( [(1, 2, 2)]_T \).

\[ P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \text{ consists of } S \text{ in column form; } Q = P^{-1} = \begin{bmatrix} 1/2 & 1 & -1/2 \\ -1 & 1 & 0 \\ 1/2 & -1 & 1/2 \end{bmatrix} \text{ is the change-of-basis matrix (found by applying ERO’s to } [P|I] \text{ until we achieve } [I|Q]. \text{ We find } [(1, 2, 2)]_T = Q \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1 \\ -1/2 \end{bmatrix}. \]