Math 254 Exam 4 Solutions

1. Consider the vector space $\mathbb{R}^2$, and set $u = (1, 2), v = (-2, 0)$. Determine whether or not $\{u, v\}$ is dependent (justify your answer).

   (I) Two vectors are dependent precisely when one is a scalar multiple of the other. If $ku = v$ then $(k, 2k) = (-2, 0)$ and hence $k = -2, 2k = 0$; this has no solutions. Thus this set is independent.

   (II) $\{u, v\}$ is spanning by the solution to the next question. $\mathbb{R}^2$ has dimension 2, hence any spanning set of size 2 must be a basis. Hence $\{u, v\}$ is a basis, and independent.

   (III) $au + bv = (a + 2a) + (-2b, 0) = (a - 2b, 2a)$. If this equals $(0, 0)$, then $a - 2b = 0, 2a = 0$. The only solution to this system is $a = b = 0$; hence this set is independent.

2. Consider the vector space $\mathbb{R}^2$, and set $u = (1, 2), v = (-2, 0)$. Determine whether or not $\{u, v\}$ is a spanning set (justify your answer).

   (I) $\{u, v\}$ is independent by the solution to the previous question. $\mathbb{R}^2$ has dimension 2, hence any independent set of size 2 must be a basis. Hence $\{u, v\}$ is a basis, and spanning.

   (II) Let (x, y) be any element of $\mathbb{R}^2$. We want to find scalars a, b such that $au + bv = (x, y)$. A calculation (not shown) yields $a = y/2, b = y/4 - x/2$. To verify: $au + bv = (y/2)(1, 2) + (y/4 - x/2)(-2, 0) = (y/2, y) + (-y/2, x) = (x, y)$. Hence some linear combination of $\{u, v\}$ will yield an arbitrary element of the vector space; hence $\{u, v\}$ is spanning.

3. Set $U = \{(a, b, c) : a + b = 2c;\ a, b, c \text{ are real}\}$. U is a subset of $\mathbb{R}^3$. Give three vectors from U, and determine whether or not U is a vector space.

   There are many vectors in U; for example, $(0, 0, 0), (-2, 2, 0), (3, 4, 7/2), (0, -2, -1), (\pi, \sqrt{2}, (\pi + \sqrt{2})/2)$.

   Because U is a subset of $\mathbb{R}^3$, we need only check three axioms for U to be a subspace (and thus a vector space in its own right).

   (Z) $(0, 0, 0)$ satisfies $0 + 0 = 2 \times 0$, hence $(0, 0, 0)$ is in U. YES

   (VA) Let $(a, b, c), (a', b', c')$ both be in U. Hence $a + b = 2c, a' + b' = 2c'$. We add the vectors to get $(a, b, c) + (a', b', c') = (a + a', b + b', c + c')$.

This will be in U if $(a+a')+(b+b')=2(c+c')$. But this must be true, by adding the two equations we had previously. YES

(SM) Let (a,b,c) be in U, and k be in $\mathbb{R}$. We have $a+b=2c$. Performing scalar multiplication, we get $k(a,b,c)=(ka,kb,kc)$. This will be in U if $(ka)+(kb)=2(kc)$. But this must be true, by multiplying the equation we had previously by k. YES

Because all three properties hold, U is a vector space.

4. Set $V=\mathbb{R}^5$. Give any two subspaces U_1,U_2 such that $U_1\oplus U_2=V$.

   There are many solutions; here is the general solution: Pick any basis S for V; it will have five elements since V has dimension 5. Split the basis into two parts S_1 and S_2. Since there were five elements in S, these were split either 4-1 or 3-2. Set $U_1=span(S_1),U_2=span(S_2)$.

   Specific solution 1: $U_1=\{(a,b,0,0,0):a,b \text{ in } \mathbb{R}\},U_2=\{(0,0,c,d,e):c,d,e \text{ in } \mathbb{R}\}$. We see that U_1,U_2 have intersection only in $\bar{0}$, and that $U_1+U_2=V$.

   Specific solution 2: $U_1=span(\{e_1,e_2,e_3,e_4\}),U_2=span(e_5)$, where the e_i are the standard basis vectors. We have U_1,U_2 intersect only in $\bar{0}$, and that $U_1+U_2=V$.

5. There are eleven properties ("axioms") one needs to check for V to be a vector space. Carefully state eight of them.

   It is not important whether you named them (A1) versus (A2) versus (M2). It is important to distinguish between the eleven axioms and the other properties that don't help prove V is a vector space (like the theorem properties). It is also important to quantify the axioms correctly:

   Correct: For any u in V, there exists $-u$ in V, such that $u+(-u)=\bar{0}$.

   Incorrect: There exists $-u$ in V such that $u+(-u)=\bar{0}$.

   Incorrect: $u+(-u)=0$.

   Incorrect: For any u in V, $u+(-u)=0$.

   Incorrect: For any u in V, for any $-u$ in V, $u+(-u)=\bar{0}$.

   Incorrect: There exists $-u$ in V, for any u in V, such that $u+(-u)=\bar{0}$.