Math 254 Exam 3 Solutions

1. Carefully define the term "dimension" as it applies to vector spaces. Give two examples: a four-dimensional vector space, and an infinite-dimensional vector space.

   The dimension of a vector space is the size of any basis of that vector space. $\mathbb{R}^4$, the set of all 4-vectors, is a four-dimensional vector space. $\mathbb{R}[x]$, the set of all polynomials with real coefficients, is an infinite-dimensional vector space.

2. Find the LU decomposition of $A = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 3 & 2 \\ -1 & 3 & 0 \end{bmatrix}$, if it exists.

   BONUS: Find the LDU decomposition of A, if it exists.

   We put A in echelon form with ERO's; we get $E_3E_2E_1A = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 6 \\ 0 & 0 & -8 \end{bmatrix}$,

   where $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, and $E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$.

   We then have $A = E_1^{-1}E_2^{-1}E_3^{-1}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 6 \\ 0 & 0 & -8 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & 6 \\ 0 & 0 & -8 \end{bmatrix}}_{U}$.

   For LDU decomposition, $D = \operatorname{diag}(1, 3, -8)$ is the diagonal of this first U; hence we factor $1, 3, -8$ out of each row, respectively. This gives

   $$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}}_{L}\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -8 \end{bmatrix}}_{D}\underbrace{\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}_{U}$$

3. Find $\begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1}$, if it exists.

   We form the augmented matrix $\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]$ and perform

ERO's to put the left side into row echelon form. We have

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] =$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -2 \\ 0 & 1 & 0 & 1 & 1 & -2 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right].\text{ Hence } \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -2 \\ 1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}.$$

The remaining problems both concern $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$.

4. Write B as the product of elementary matrices.

   We first put B into diagonal form using elementary matrices. One way:

$$\begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}}_{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.\text{ Hence}$$

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}}_{B} = \begin{bmatrix} 1 & 0 \\ -\frac{3}{2} & 1 \end{bmatrix}^{-1} \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & -\frac{3}{2} \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & 0 \\ \frac{3}{2} & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & \frac{3}{2} \\ 0 & 1 \end{bmatrix}.$$

5. Calculate $f(B)$, for the polynomial $f(x) = x^3 + 2x^2 - 3I_2$.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}, B^2 = \begin{bmatrix} 13 & 18 \\ 18 & 25 \end{bmatrix}, B^3 = \begin{bmatrix} 80 & 111 \\ 111 & 154 \end{bmatrix}$$

$$f(B) = \begin{bmatrix} 80 & 111 \\ 111 & 154 \end{bmatrix} + 2\begin{bmatrix} 13 & 18 \\ 18 & 25 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 103 & 147 \\ 147 & 201 \end{bmatrix}.$$

   This problem illustrates a nice feature of symmetric matrices; if you do arithmetic (addition, multiplication, etc.) with them, the results will remain symmetric.