1. Carefully define the following terms: Proof by Cases Theorem, Nonconstructive Existence Theorem, Proof by Reindexed Induction

The Proof by Cases theorem says: Let \( p, q \) be propositions. Suppose there are propositions \( c_1, c_2, \ldots, c_k \) with \( c_1 \lor c_2 \land \cdots \lor c_k \equiv T \). Now, if \((p \land c_1) \rightarrow q, (p \land c_2) \rightarrow q, \ldots, (p \land c_k) \rightarrow q\) are all true, then \( p \rightarrow q \) is true. The Nonconstructive Existence Theorem says: If \( \forall x \in D, \neg P(x) \equiv F \), then \( \exists x \in D \) \( P(x) \) is true. To prove \( \forall x \in \mathbb{N}, P(x) \) by reindexed induction, we must (a) prove \( P(1) \) is true; and (b) prove \( \forall x \in \mathbb{N} \) with \( x \geq 2 \), \( P(x-1) \rightarrow P(x) \).

2. Carefully define the following terms: Proof by Minimum Element Induction Thm, well-ordered, big \( O \)

The Proof by Minimum Element Induction Theorem says: Let \( S \) be a nonempty set of integers. If \( S \) has a lower bound, then it has a minimum. Let \( S \) be a set of numbers with some ordering \( < \). We say that \( S \) is well-ordered by \( < \) if every nonempty subset of \( S \) has a minimum according to \( < \). Let \( a_n \) and \( b_n \) be sequences. We say that \( a_n = O(b_n) \) if \( \exists n_0 \in \mathbb{N}, \exists M \in \mathbb{R}, \forall n \geq n_0, |a_n| \leq M|b_n| \).

3. Let \( x \in \mathbb{R} \). Use cases to prove that \(|x - 2| + |x - 5| \geq 3\).

Case 1: If \( x < 2 \), then \(|x - 2| + |x - 5| = 2 - x + 5 - x = 7 - 2x \). We multiply \( x < 2 \) by \(-2\) to get \(-2x > -4\), and add \( 7 \) to get \( 7 - 2x > 7 - 4 = 3 \). Hence \(|x - 2| + |x - 5| = 7 - 2x > 3\).

Case 2: If \( 2 \leq x \leq 5 \), then \(|x - 2| + |x - 5| = x - 2 + 5 - x = 3 \geq 3\).

Case 3: If \( x > 5 \), then \(|x - 2| + |x - 5| = x - 2 + x - 5 = 2x - 7 \). We multiply \( x > 5 \) by \( 2\) to get \( 2x > 10 \), and add \(-7\) to get \( 2x - 7 > 10 - 7 = 3 \). Hence, \(|x - 2| + |x - 5| = 2x - 7 > 3\).

In all three cases, the desired inequality holds.

4. Prove that \( \forall x \in \mathbb{R}, [-x] = -[x] \).

Let \( x \in \mathbb{R} \) be arbitrary. Applying the definitions of ceiling and floor, we get \([-x] \leq -x < [-x] + 1 \) and \([x] - 1 < x \leq [x] \). We multiply the latter by \(-1\) to get \(-[x] + 1 > -x \geq -[x] \). We now have a choice for how to continue.

SOLUTION 1: We combine inequalities to get \([-x] \leq -x < -[-x] + 1 \) and \([-x] \leq -x < [-x] + 1 \). Hence \( -[x] - 1 < [-x] < -[x] + 1 \). Since these are integers, by Thm. 1.12, we have \([-x] = -[x] \).

SOLUTION 2: Both \([-x]\) and \(-[x]\) are integers satisfying \( n \leq -x < n + 1 \). Because the floor of any real \( x \) is unique (by a theorem proved in class, part of the definition of “floor”), these integers must be equal.

5. Use induction to prove that for all \( n \in \mathbb{N}, \binom{2n}{n} \leq 4^n \).

Proof by vanilla induction. Base case \( n = 1 \): \( \binom{2}{1} = \binom{2}{1} = \frac{2!}{1!1!} = 2 \), which is less than \( 4^1 = 4 \).
Now, let \( n \in \mathbb{N} \) and suppose that \( \binom{2n}{n} \leq 4^n \). We have \( \frac{2^{n+1}}{(n+1)!} = \binom{2n}{n} < \frac{2^{n+2}(2n+1)}{(n+1)(n+2)!} \binom{2n}{n} = \frac{2^{n+1}(n+1)!}{(n+1)!} \) and \( 4^n = 4^n \). Hence

\[
\binom{2(n+1)}{n+1} \leq 4^{n+1}.
\]

6. Solve the recurrence with initial conditions \( a_0 = 1, a_1 = 4 \) and relation \( a_n = 3a_{n-1} - 2a_{n-2} \) \((n \geq 2)\).

Our characteristic polynomial is \( r^2 - 3r + 2 = (r - 2)(r - 1) \), so the general solution is \( a_n = A2^n + B1^n = A2^n + B \). We now apply the initial conditions to get \( 1 = a_0 = A2^0 + B = A + B \) and \( 4 = a_1 = A2^1 + B = 2A + B \). We now solve the system \( (A + B = 1, 2A + B = 4) \) to get \( A = 3, B = -2 \). Hence the specific solution is \( a_n = 3 \cdot 2^n - 2 \).

7. Consider the nonstandard order \( \prec \) on \( \mathbb{Z} \) given by \( 0 \prec 1 \prec -1 \prec 2 \prec -2 \prec 3 \prec \cdots \). The smallest element is 0, the second smallest is 1. Find a formula for the \( n \)th smallest element.

The \( n \)th smallest element is \( \begin{cases} n/2 & \text{if } n \text{ is even} \\ (1 - n)/2 & \text{if } n \text{ is odd} \end{cases} \).

One can avoid cases, at the expense of a messier formula, e.g. \( \lceil \frac{n(-1)^n}{2} \rceil \) or \( (-1)^n \lfloor \frac{n}{2} \rfloor \).

8. Consider the sequence \( a_n = 3n^2 + 100n + 1 \). Prove that \( a_n = \Theta(n^2) \).

We need to prove both \( a_n = O(n^2) \) (harder) and \( a_n = \Omega(n^2) \) (easier).

\( a_n = O(n^2) \): Take \( n_0 = 100, M = 5 \). Let \( n \geq n_0 = 100 \). We have \( |a_n| = |3n^2 + 100n + 1| = 3n^2 + 100n + 1 \leq 3n^2 + n^2 + n^2 = 5n^2 = M|b_n| \).

\( a_n = \Omega(n^2) \): Take \( n_0 = 1, M = 1 \). Let \( n \geq n_0 = 1 \). We have \( M|a_n| = 1|3n^2 + 100n + 1| = 3n^2 + 100n + 1 \geq n^2 = |b_n| \).

9. Prove that \( \forall n \in \mathbb{N}_0 \), the Fibonacci numbers \( F_n \) satisfy \( F_n < 1.9^n \).

Proof by strong induction. We need two base cases: \( F_0 = 0 < 1 = 1.9^0 \), and \( F_1 = 1 < 1.9^1 \).

Now, let \( n \in \mathbb{N}_0 \), and assume that \( F_n < 1.9^n \) and \( F_{n+1} < 1.9^{n+1} \). We have \( F_{n+2} = F_{n+1} + F_n < 1.9^{n+1} + 1.9^n = 1.9^n(1.9 + 1) = 1.9^n(2.9) < 1.9^n(3.61) = 1.9^n(1.9)^2 < 1.9^{n+2} \). Hence \( F_{n+2} < 1.9^{n+2} \).

10. Find a recurrence relation for sequence \( T_n \) such that the Master Theorem would give \( T_n = \Theta(\sqrt{n} \log n) \). Describe an algorithm that would satisfy your recurrence relation.

The form of the solution can only arise in the “middle \( c_n \)” case. Hence, \( k = d = 1/2 \). Hence \( a = 1/2 \) and \( c_n = \Theta(\sqrt{n}) \). One possibility is \( a = 2, b = 4, c_n = \sqrt{n} \), which gives \( T_n = 2T_{n/4} + \sqrt{n} \). An algorithm that satisfies this recurrence relation would divide the size \( n \) problem into four parts, call itself recursively on two of the parts, and have an overhead of \( \sqrt{n} \) putting those two results together.