1. Carefully define the following terms: = (for sets), union, disjoint.

Two sets are equal if they contain exactly the same elements. Given sets \( S, T \), their union is the set \( \{ x : x \in S \lor x \in T \} \). Two sets are disjoint if their intersection is equal to the empty set.

2. Carefully define the following terms: De Morgan’s Law (for sets), Cantor’s Theorem, transitive

Given sets \( S, T, U \) with \( S \subseteq U \) and \( T \subseteq U \), De Morgan’s law states that (a) \( (S \cup T)^c = S^c \cap T^c \); and (b) \( (S \cap T)^c = S^c \cup T^c \). Cantor’s Theorem states that no set is equicardinal with its power set. Given a relation \( R \) on set \( S \), we say that \( R \) is transitive if for all \( x, y, z \in S \), \( (xRy \land yRz) \rightarrow xRz \).

3. Let \( R, S \) be sets with \( R \setminus S = S \setminus R \). Prove that \( R \subseteq S \).

Let \( x \in R \) be arbitrary. We will prove that \( x \in S \) by contradiction; that is, assume that \( x \notin S \). By conjunction, \( (x \in R) \land (x \notin S) \). Hence, \( x \in R \setminus S \). Because \( R \setminus S = S \setminus R \), in fact \( x \in S \setminus R \). Hence \( x \in S \land x \notin R \). By simplification, \( x \notin R \). This is a contradiction. Hence, \( x \in S \).

4. Prove or disprove: For all sets \( R, S, T \) satisfying \( R \subseteq S \), \( S \subseteq T \), and \( T \subseteq R \), we must have \( R = S \).

The statement is true. We have \( R \subseteq S \) by hypothesis, so it suffices to prove that \( S \subseteq R \). Let \( x \in S \) be arbitrary. Because \( S \subseteq T \), we have \( x \in T \). Because \( T \subseteq R \), \( x \in R \).

5. Prove or disprove: For all sets \( R, S \), we have \( R \times S = S \times R \).

The statement is false; to disprove, we need explicit examples for \( R, S \). One possible answer is \( R = \{ a \}, S = \{ b \} \). To prove that \( R \times S \neq S \times R \), we need an explicit element that is in one set but not the other. \( (a, b) \in R \times S \), but \( (a, b) \notin S \times R = \{(b, a)\} \).

6. Prove or disprove: For all sets \( R, S \), we have \(|R \times S| = |S \times R|\).

Note: The theorem \(|R \times S| = |R| \cdot |S|\) holds only for finite sets \( R, S \) and will only provide partial credit.

To prove two sets are equicardinal, we need an explicit pairing between their elements. The natural one is \( (x, y) \leftrightarrow (y, x) \), for every \( x \in R \) and \( y \in S \).

7. Consider relation \( R = \{(a, b) : a^2 \geq b\} \) on \( \mathbb{Q} \). Prove or disprove that \( R \) is reflexive.

The statement is false; to disprove, we need an explicit counterexample. If we take \( a = b = \frac{1}{2} \), we see that \( a^2 = \frac{1}{4} \ngeq \frac{1}{2} = b \), so \( (\frac{1}{2}, \frac{1}{2}) \notin R \) and hence \( R \) is not reflexive.
8. Prove or disprove: For all sets $R, S$, we have $2^R \cup 2^S = 2^{R\cup S}$.

The statement is false; to disprove, we need explicit examples for $R, S$. One possible answer is $R = \{a\}, S = \{b, c\}$. To prove that $2^R \cup 2^S \neq 2^{R\cup S}$, we need an explicit element that is in one set but not the other. \{a, b\} $\in 2^{R\cup S}$, as it is a subset of $R \cup S = \{a, b, c\}$. However, \{a, b\} $\notin 2^R$, as it is not a subset of $R$. \{a, b\} $\notin 2^S$, as it is not a subset of $S$. Hence, \{a, b\} $\notin 2^R \cup 2^S$.

9. Let $R, S, T$ be sets. Prove that $R \cap (S \cup T) \subseteq (R \cap S) \cup (R \cap T)$. Your answer should not use any theorems about sets.

SOLUTION 1: Let $x \in R \cap (S \cup T)$. Hence $x \in R \land x \in (S \cup T)$. By simplification twice, we get $x \in R$ and $x \in (S \cup T)$. Hence, $x \in S \lor x \in T$. We now have two cases:

Case $x \in S$: By conjunction, $x \in R \land x \in S$. Hence, $x \in (R \cap S)$. By addition, $x \in (R \cap S) \lor x \in (R \cap T)$.

Case $x \in T$: By conjunction, $x \in R \land x \in T$. Hence, $x \in (R \cap T)$. By addition, $x \in (R \cap S) \lor x \in (R \cap T)$.

In both cases, $x \in (R \cap S) \lor x \in (R \cap T)$, and hence $x \in (R \cap S) \cup (R \cap T)$.

SOLUTION 2: Let $x \in R \cap (S \cup T)$. Hence $x \in R \land x \in (S \cup T)$. Hence $(x \in R) \land (x \in S \lor x \in T)$. Applying the distributivity theorem (for propositions), we get $(x \in R \land x \in S) \lor (x \in R \land x \in T)$. Hence $(x \in R \cap S) \lor (x \in R \cap T)$, and finally $x \in (R \cap S) \cup (R \cap T)$.

10. Consider relation $R = \{(a, b) : b \leq a \leq 3b\}$ on $\mathbb{N}_0$. Compute and simplify $R \circ R$. Your answer should not use quantifiers.

We start with $R \circ R = \{(a, c) : \exists b \in \mathbb{N}_0, aRb \land bRc\}$. This gives us four inequalities: $b \leq a \leq 3b$ and $c \leq b \leq 3c$. We combine two of them as $c \leq b \leq a$, and the other two as $a \leq 3b \leq 9c$. Hence the simplified version is $R \circ R = \{(a, c) : c \leq a \leq 9c\}$. Finding this, with justification, is enough for full credit.

For anyone curious about a proof that these sets are equal, here is an explicit calculation of $b$: Let $(a, c)$ satisfy $c \leq a \leq 9c$. If $c \leq a \leq 3c$, we take $b = c$. If instead $3c < a < 9c$, we take $b = 3c$. 
