1. Carefully define the following terms: \( \cap, \cup \) (absolute) complement, Cartesian product.

Given sets \( S, T \), \( S \cap T = \{ x : x \in S \land x \in T \} \). Given sets \( S, T, S \cup T = \{ x : x \in S \lor x \in T \} \). Given sets \( S, U \) with \( S \subseteq U \), we define the absolute complement of \( S \) as \( U \setminus S \). Given sets \( S, T \), we define the Cartesian product of \( S, T \) as \( \{(x, y) : x \in S, y \in T \} \).

2. Carefully define the following terms: relation, symmetric (relation), antisymmetric (relation), trichotomous (relation).

Given sets \( S, T \), a relation from \( S \) to \( T \) is a subset of \( S \times T \). A relation \( R \) on \( S \) is symmetric if for all \( x, y \in S \), \( xRy \leftrightarrow yRx \). A relation \( R \) on \( S \) is antisymmetric if for all \( x, y \in S \), \( (xRy \land yRx) \rightarrow x = y \). A relation \( R \) on \( S \) is trichotomous if for all \( x, y \in S \), \( x \neq y \rightarrow xRy \lor yRx \).

3. Let \( S = \{a, b\} \). Give a two-element subset of \( 2^{S \times S} \). Be careful with notation.

Note that \( S \times S = \{(a, a), (a, b), (b, a), (b, b)\} \). Elements of \( 2^{S \times S} \) are subsets of \( S \times S \). We seek a set, which contains two elements. Each of those elements must be a subset of \( S \times S \), namely a set of ordered pairs. Many solutions are possible, such as \( \{(a, a), (b, b)\} \) or \( \emptyset, S \times S \) or \( \{(a, a), (a, b), (a, a), (b, a)\} \).

4. Let \( S \) be a set. Prove that \( S \cup \emptyset = S \).

This must be proved in two parts. First we prove \( \subseteq \). Let \( x \in S \cup \emptyset \). Then \( x \in S \lor x \in \emptyset \). We have two cases: \( x \in S \) or \( x \in \emptyset \). The second case can’t happen, so \( x \in S \). This proves \( S \cup \emptyset \subseteq S \). Next, we prove \( \supseteq \). Let \( x \in S \). By addition, \( x \in S \lor x \in \emptyset \). Hence \( x \in S \cup \emptyset \). This proves \( S \cup \emptyset \supseteq S \).

5. Give a partition of \( \mathbb{Z} \) with three parts.

Many solutions are possible; all of them consist of a set of three parts such as \( \{P_0, P_1, P_2\} \). One solution is \( P_0 = \{0\}, P_1 = \mathbb{N}, P_2 = \{x \in \mathbb{Z} : x < 0\} \). Another solution is to apply the Division Algorithm with \( 3 \). \( P_i \) will be the set of integers with remainder \( i \) (which must be 0, 1, or 2). Another solution is \( P_0 = \{0\}, P_1 = \{1\}, P_2 = \mathbb{Z} \setminus \{0, 1\} \).

For problems 6 and 7, take ground set \( S = \{-1, 0, 1\} \) with relation \( R = \{(a, b) : a \leq b^2\} \).

6. With \( R, S \) as above, prove or disprove that \( R \) is reflexive.

The statement is true. Because \(-1 \leq (-1)^2 \), \((-1, -1) \in R \). Because \( 0 \leq 0^2 \), \((0, 0) \in R \). Because \( 1 \leq 1^2 \), \((1, 1) \in R \). These three together imply that \( R \) is reflexive.

7. With \( R, S \) as above, prove or disprove that \( R \) is transitive.

The statement is false. We need a specific counterexample. There is only one (it can be found by drawing the relation’s digraph). Because \( 1 \leq (-1)^2 \), \((-1, 1) \in R \). Because \(-1 \leq 0^2 \), \((-1, 0) \in R \). However, \((1, 0) \notin R \), because \( 1 \not\leq 0^2 \). Hence \( R \) is not transitive.

8. Prove or disprove: For all sets \( R, S \), we have \( R \setminus S = R \Delta S \).

The statement is false. We need a specific counterexample. Many are possible. A simple one is \( R = \{1, 3\}, S = \{2, 3\} \). We have \( R \setminus S = \{1\} \), while \( R \Delta S = (R \setminus S) \cup (S \setminus R) = \{1, 2\} \).

9. Prove or disprove: For all sets \( R, S, T \) satisfying \( R \subseteq S \), \( S \subseteq T \), and \( T \subseteq R \), we must have \( R = S \).

The statement is true. To prove \( R = S \), we need to prove \( R \subseteq S \) (one of our hypotheses already), and \( S \subseteq R \). Let \( x \in S \). Since \( S \subseteq T \), \( x \in T \). Since \( T \subseteq R \), \( x \in R \). Hence \( S \subseteq R \).

10. Prove or disprove: \(|\mathbb{N}| = |\mathbb{N}_0 \times \mathbb{N}_0| \).

The statement is true.

PROOF 1: As in Thm 9.17 and Exercise 9.24, for any \( n \in \mathbb{N} \) we can uniquely write \( n = 2^a(2b + 1) \), and pair \( n \leftrightarrow (a, b) \).

PROOF 2: We write all the ordered pairs in \( \mathbb{N}_0 \times \mathbb{N}_0 \) in the first quadrant at their locations, and take a zig-zag path starting at the origin and passing through all the pairs. We pair the \( n^{th} \) position along the path with the ordered pair at that position.

PROOF 3: We pair \( \mathbb{N} \) with a subset of \( \mathbb{N}_0 \times \mathbb{N}_0 \), for example via \( n \leftrightarrow (n, 0) \). This proves that \( |\mathbb{N}| \leq |\mathbb{N}_0 \times \mathbb{N}_0| \).

We next pair \( \mathbb{N}_0 \times \mathbb{N}_0 \) with a subset of \( \mathbb{N} \), for example via \( (a, b) \leftrightarrow 2^a3^b \). This proves that \( |\mathbb{N}| \geq |\mathbb{N}_0 \times \mathbb{N}_0| \).

Lastly, we apply the Cantor-Schröder-Bernstein Theorem.