1. Carefully define the following terms: free variable, predicate, counterexample, Left-to-Right Principle.

A free variable is a variable that has not been bound, drawn from some domain. A predicate is a collection of propositions, indexed by one or more free variables, each drawn from its domain. A counterexample is a specific domain element chosen to make a predicate false. The Left-to-Right Principle states that variables are bound from left to right.

2. Carefully define the following terms: Uniqueness Proof Theorem, Proof by Contradiction Theorem, Proof by Induction Theorem, well-ordered set.

The Uniqueness Proof theorem states: there is at most one domain element satisfying predicate $P$ if $\forall x, y \in D, P(x) \land P(y) \rightarrow x = y$. The Proof by Contradiction theorem states: For propositions $p, q$, if $(p \land \neg q) \equiv F$, then $p \rightarrow q$ is true. The Proof by Induction theorem states: To prove $\forall x \in \mathbb{N}, P(x)$, we prove (a) $P(1)$ is true; and (b) $\forall x \in \mathbb{N}, P(x) \rightarrow P(x+1)$.

3. Simplify $\neg(\exists x, \forall y, \exists z, (x < y) \rightarrow (x < z))$ as much as possible (i.e. where nothing is negated). Do not prove or disprove this statement.

$\forall x, \exists y, \exists z, (x < y) \land (x \geq z)$, or (nicer) $\forall x, \exists y, \exists z, x \leq y$.

4. Recall that $\mathbb{R} \setminus \mathbb{Q}$ is the set of irrational numbers. Let $a \in \mathbb{R} \setminus \mathbb{Q}$, $b \in \mathbb{Q}$. Use proof by contradiction to prove that $a + b \notin \mathbb{Q}$.

Because $b \in \mathbb{Q}$, there are integers $m, n$ with $n \neq 0$ and $b = \frac{m}{n}$. Now, assume by way of contradiction that $a + b \in \mathbb{Q}$. Then there are integers $s, t$ with $t \neq 0$ and $a + b = \frac{s}{t}$. We calculate $a = (a + b) - b = \frac{s}{t} - \frac{m}{n} = \frac{sn - mt}{nt}$. Now, $sn - mt$, $nt$ are integers, and $nt \neq 0$, so $a \in \mathbb{Q}$. This is a contradiction.

5. Prove or disprove: $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, (x < y) \rightarrow [x] \leq [y]$.

The statement is false. To prove this, we need to prove $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, (x < y) \land ([x] > [y])$. Take $x = 0.3, y = 0.4$. We have $x < y$ but $[x] = 1 > 0 = [y]$. Many other $x, y$ are possible.

6. Let $n \in \mathbb{Z}$. Use the Division Algorithm to prove that $\frac{(n-1)(n+2)}{2} \in \mathbb{Z}$.

Apply DA to get integers $q, r$ with $n = 2q + r$, and $0 \leq r < 2$. We now have two cases. If $r = 0$, then $\frac{(n-1)(n+2)}{2} = (n-1)q \in \mathbb{Z}$. If instead $r = 1$, then $\frac{(n-1)(n+2)}{2} = (2q+1-1)(n+2) = q(n+2) \in \mathbb{Z}$.

7. Recall the Fibonacci numbers given by $F_0 = 0, F_1 = 1, F_n = F_{n-1} + F_{n-2}$ (for $n \geq 2$). Prove that for all $n \in \mathbb{N}_0$, $F_{n+2} = 1 + \sum_{i=0}^{n} F_i$.

Base case $n = 0$: $F_2 = 1 = 1 + F_0 + F_1$. Inductive case: Let $n \in \mathbb{N}_0$ and assume that $F_{n+2} = 1 + \sum_{i=0}^{n} F_i$. Add $F_{n+1}$ to both sides: $F_{n+3} = F_{n+1} + F_{n+2} = F_{n+1} + 1 + \sum_{i=0}^{n} F_i = 1 + \sum_{i=0}^{n+1} F_i$. Hence $F_{n+3} = 1 + \sum_{i=0}^{n+1} F_i$.

8. Let $x \in \mathbb{R}$. Prove that $[x]$ exists. That is, prove $\exists n \in \mathbb{Z}$, $n \leq x < n + 1$.

Let $S$ be the set of all integers less than or equal to $x$. This is a nonempty set, with an upper bound $(x)$, so by the Maximum Element Induction theorem, there is some maximum element $n \in S$. Since $n \in S$, $n \geq x$. We now prove $x < n + 1$. Assume, by way of contradiction, that $x \geq n + 1$. But then $n + 1 \in S$, and $n + 1 > n$, a contradiction since $n$ was a maximum. Hence $n \leq x < n + 1$.

9. Use induction to prove $\forall n \in \mathbb{N}, \frac{(2n)!}{n!n!} \geq 2^n$.

Base case $n = 1$: $\frac{2!}{1!1!} = 2 \geq 2^1$. Inductive case: Let $n \in \mathbb{N}$, and assume that $\frac{(2n)!}{n!n!} \geq 2^n$. Multiply both sides by $\frac{(2(n+1))!}{(n+1)!(n+1)!}$ to get $\frac{(2(n+1))!}{(n+1)!(n+1)!} \geq 2^n \cdot \frac{(2n)!}{n!n!} \cdot \frac{1}{(n+1)!(n+1)!} \geq 2^n \cdot \frac{(2n+2)!}{n!n!} \cdot \frac{1}{(n+1)!(n+1)!} = 2^n \cdot \frac{(2n+2)!}{n!n!} \cdot \frac{1}{(n+1)!(n+1)!} \geq 2^{n+1}$.

Hence $\frac{(2(n+1))!}{(n+1)!(n+1)!} \geq 2^{n+1}$.

10. Let $\mathbb{R}^+$ denote the positive real numbers. Prove that $\forall a \in \mathbb{R}^+, \exists b \in \mathbb{R}^+, \forall x \in \mathbb{R}, |x - 2| < b \rightarrow |3x - 6| < a$.

Let $a \in \mathbb{R}^+$ be arbitrary. Choose $b = \frac{a}{3}$. Now, let $x \in \mathbb{R}$ with $|x - 2| < b$. We have $|x - 2| < b = \frac{a}{3}$. Multiplying both sides by 3, we get $|3x - 6| = 3|x - 2| < a$. Hence $|3x - 6| < a$. This proves that $\lim_{x \to 2} 3x = 6$; to learn much more like this, take Math 534A.