1. Carefully define the following terms: composite, conjunction, tautology, Double Negation semantic theorem

Let \( n \in \mathbb{Z} \) with \( n \geq 2 \). We call \( n \) composite if there is some \( a \in \mathbb{Z} \) such that \( 1 < a < n \) and \( a|n \). Let \( p, q \) be propositions. Their conjunction is the proposition that is true if \( p, q \) are both true, and false otherwise. A tautology is a (compound) proposition that is always true. The Double Negation semantic theorem states that for every proposition \( p \), we have \( \neg(\neg p) \equiv p \).

2. Carefully define the following terms: Addition semantic theorem, Trivial Proof theorem, Direct Proof, converse.

The Addition semantic theorem states that for any propositions \( p, q \), we have \( p \vdash p \lor q \). The Trivial Proof theorem states that for any propositions \( p, q \), we have \( q \vdash p \rightarrow q \). The Direct Proof theorem states that for any propositions \( p, q \), if \( p \vdash q \) is valid, then \( p \rightarrow q \) is true. The converse of conditional proposition \( p \rightarrow q \) is \( q \rightarrow p \).

3. Calculate and simplify \( \frac{(13.9) + (-1.2)!}{(8.4)!} \).

We have \( \frac{(13.9) + (-1.2)!}{(8.4)!} = \frac{(13.9)!}{8.4!} = \frac{11!}{8!} = \frac{11 \cdot 10 \cdot 9!}{8!} = 11 \cdot 10 = 110 \).

4. Let \( a, b, c \in \mathbb{Z} \). Suppose that \( a|b \) and \( a|c \). Prove that \( a|(b + c) \).

Because \( a|b \) there is some \( m \in \mathbb{Z} \) with \( b = ma \). Because \( a|c \) there is some \( n \in \mathbb{Z} \) with \( c = na \). Adding, we get \( b + c = ma + na = (m + n)a \). Now \( a|(b + c) \) because \( m + n \in \mathbb{Z} \).

5. Use truth tables to prove the half of De Morgan’s Law which states that for any propositions \( p, q \) we have \( \neg(p \lor q) \equiv (\neg p) \land (\neg q) \).

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The fourth and seventh columns, as highlighted, agree. Hence \( \neg(p \lor q) \equiv (\neg p) \land (\neg q) \).

6. Simplify \( \neg((p \rightarrow q) \land r) \) as much as possible. (i.e. where only basic propositions are negated)

We start with \( \neg((p \rightarrow q) \land r) \). Applying conditional interpretation, this is equivalent to \( \neg((q \lor \neg p) \land r) \). Applying De Morgan’s Law, this is equivalent to \( (\neg(q \lor \neg p)) \lor (\neg r) \). Applying De Morgan’s Law again, this is equivalent to \( (\neg q) \land (\neg r) \land (\neg r) \). Finally, applying double negation, this is equivalent to \( (\neg q) \land p \lor (\neg r) \).

7. Let \( x \in \mathbb{R} \). Prove that if \( x \) is irrational then \( \frac{x}{3} \) is irrational.

We use a contrapositive proof. Assume that \( \frac{x}{3} \) is rational. Then there are integers \( m, n \), with \( n \neq 0 \), such that \( \frac{x}{3} = \frac{m}{n} \). Multiplying both sides by 3 we get \( x = \frac{3m}{n} \). Now, \( 3m, n \) are integers with \( n \neq 0 \), so \( x \) is rational.

8. Let \( n \in \mathbb{Z} \). Suppose that \( n \) is even. Prove that \( 3n^2 + 1 \) is odd.

We use a direct proof. Suppose that \( n \) is even. Then there is an integer \( m \) with \( n = 2m \). Now, \( 3n^2 + 1 = 3(2m)^2 + 1 = 3(4m^2) + 1 = 2(6m^2) + 1 \). Because \( 6m^2 \) is an integer, \( 3n^2 + 1 \) is odd.

9. Using semantic theorems, prove that for any propositions \( p, q, r \), we have \( ((p \lor q) \lor r), (\neg q) \vdash r \lor p \).

Start with hypothesis \( (p \lor q) \lor r \). Applying commutativity of \( \lor \), we get \( (q \lor p) \lor r \). Applying associativity of \( \lor \), we get \( q \lor (p \lor r) \). Now apply disjunctive syllogism to this and to hypothesis \( \neg q \) to get \( p \lor r \).

10. Using semantic theorems, prove that for any propositions \( p, q, r \), we have \( (p \rightarrow q), (q \rightarrow r) \vdash (p \rightarrow r) \).

If \( q \) is true, then applying modus ponens to hypothesis \( q \rightarrow r \) gives \( r \). Applying addition gives \( r \lor \neg p \).

If instead \( q \) is false, then applying modus tollens to hypothesis \( p \rightarrow q \) gives \( \neg p \). Applying addition gives \( r \lor \neg p \).

Either way we have \( r \lor \neg p \). Applying conditional interpretation we get \( p \rightarrow r \).