MATH 245 F20, Exam 1 Solutions

1. Let \( b, c \) be odd integers. Without using theorems, prove that \( b(c - 2) \) is odd.
   Since \( b, c \) are odd, there exist integers \( y, z \) with \( b = 2y + 1, c = 2z + 1 \). We calculate \( b(c - 2) = (2y + 1)(2z - 1) = 4yz - 2y + 2z - 1 = 4yz - 2y + 2z - 2 + 1 = 2(2yz - y + z - 1) + 1 \). Since \( y, z \) are integers, so is \( 2yz - y + z - 1 \). Hence \( b(c - 2) \) is odd, being the sum of 1 with twice an integer.

2. Prove or disprove: For all propositions \( p, q \), the proposition \((p \uparrow q) \downarrow (p \leftrightarrow q)\) is a contradiction.
   We look at the truth table at right, and see that the last column is all \( F \). Hence \((p \uparrow q) \downarrow (p \leftrightarrow q) \equiv F\), and therefore \((p \uparrow q) \downarrow (p \leftrightarrow q)\) is a contradiction.

3. Let \( p, q, r, s \) be propositions. Prove that \( p \lor q, q \land r, p \rightarrow s \rightarrow q \lor s \).
   We begin by assuming that \( p \lor q, q \land r \), and \( p \rightarrow s \) are all true.
   SOLUTION 1: We only need the hypothesis \( q \land r \). By simplification, \( q \). By addition, \( q \lor s \).
   SOLUTION 2: We have two cases, based on \( p \lor q \). Case 1: If \( p \) is true, we apply modus ponens to \( p \rightarrow s \) to get \( s \). By addition, \( q \lor s \). Case 2: If instead \( q \) is true, we directly apply addition to get \( q \lor s \). In both cases \( q \lor s \) holds.
   SOLUTION 3: It is also possible to do this with a huge truth table (16 rows!). NOT RECOMMENDED

4. Prove the following without truth tables: For any propositions \( p, q, r, s \), we have \( p \rightarrow q, q \rightarrow r, r \rightarrow s \rightarrow p \rightarrow s \).
   We begin by assuming that \( p \rightarrow q, q \rightarrow r, r \rightarrow s \) are all true.
   We consider two cases: \( q \) might be \( T \) or \( F \). If \( q \) is \( T \), then by modus tollens with \( p \rightarrow q \), we have \( \neg p \).
   By addition, \( s \lor \neg p \). If instead \( q \) is \( F \), then by modus ponens with \( q \rightarrow r \), we have \( q \). By modus ponens with \( r \rightarrow s \), we have \( s \). By addition, \( s \lor \neg p \).
   In both cases, we get \( s \lor \neg p \). Finally, by conditional interpretation, we get \( p \rightarrow s \).
   It is also possible to do this using different cases, such as \( p \) being \( T \) or \( F \).

5. Let \( x \in \mathbb{R} \). Prove that if \( x^2 \) is irrational, then \( x \) is irrational.
   We use a contrapositive proof. Assume that \( x \) is rational. Then there are integers \( a, b \) with \( b \neq 0 \) and \( x = \frac{a}{b} \). We have \( x^2 = \frac{a^2}{b^2} \). Note that \( a^2, b^2 \) are integers (since \( a, b \) are), and \( b^2 \neq 0 \) (since \( b \neq 0 \)). Hence \( x^2 \) is rational.

6. Fix our domain to be \( \mathbb{Z} \) for all variables. Simplify the following proposition as much as possible (where nothing is negated): \( \neg \forall x \\forall y \exists z \ (x < y) \rightarrow (x < z \leq y) \).
   We first pull the negation inside the quantifiers: \( \exists x \exists y \forall z \ (x < y) \land \neg(x < z \leq y) \).
   We now apply Theorem 2.16 to get: \( \exists x \exists y \forall z \ (x < y) \land \neg((x < z) \land (z \leq y)) \).
   We interpret the double inequality (see p.11) to get: \( \exists x \exists y \forall z \ (x < y) \land \neg((x < z) \land (z \leq y)) \).
   We apply De Morgan’s Law (for propositions) to get: \( \exists x \exists y \forall z \ (x < y) \land (((x < z) \lor (z > y)) \lor (x < z)) \).
   We now simplify to get our answer: \( \exists x \exists y \forall z \ (x < y) \land (x \geq z) \lor (z > y) \).
   NOTE: \( (x \geq z) \lor (z > y) \) cannot be combined to a double inequality, but it is possible to use distributivity to get the alternative answer \( \exists x \exists y \forall z \ (z \leq x < y) \lor (x < y < z) \).

7. Prove or disprove this proposition: \( \forall x \in \mathbb{Z}, \exists y \in \mathbb{Z}, \ (x \neq y) \land (y|x) \).
   The statement is true, and we will supply a direct proof. Let \( x \in \mathbb{Z} \) be arbitrary. We have two cases, based on whether \( x = 0 \). NOTE: it is not possible to pick a single \( y \) that works for every \( x \).
   If \( x = 0 \), choose \( y = 5 \). We have \( x \neq y \) and \( 0 = (0)(5) \), so \( y|x \).
   If \( x \neq 0 \), choose \( y = -x \). We have \( x \neq y \), since otherwise \( x = y = -x \) and so \( x = -x \) but \( x \neq 0 \). Also \( x = (−1)(y) \), so \( y|x \).