1. Carefully define the following terms: Proof by Contradiction theorem, Proof by Cases theorem, Proof by Induction, Proof by Reindexed Induction.

Let \( p, q \) be propositions. The Proof by Contradiction theorem tells us that if \( p \land \neg q \equiv F \), then \( p \rightarrow q \) is true. Let \( p, q \) be propositions. The Proof by Cases theorem tells us that if there are propositions \( c_1, c_2, \ldots, c_k \) with \( c_1 \lor c_2 \lor \cdots \lor c_k \equiv T \), and each of \( (p \land c_1) \rightarrow q, (p \land c_2) \rightarrow q, \ldots, (p \land c_k) \rightarrow q \), then \( p \rightarrow q \) is true. To prove \( \forall x \in \mathbb{N} \ P(x) \) by induction, we must (a) Prove \( P(1) \); and (b) Prove \( \forall x \in \mathbb{N}, P(x) \rightarrow P(x+1) \). To prove \( \forall x \in \mathbb{N} \ P(x) \) by reindexed induction, we must (a) Prove \( P(1) \); and (b) Prove \( \forall x \in \mathbb{N} \) with \( x \geq 2 \), \( P(x-1) \rightarrow P(x) \).

2. Carefully define the following terms: well-ordered, recurrence, big Omega, big Theta.

Let \( S \) be a set of numbers, with an ordering \( < \). We say that \( S \) is well-ordered by \( < \) if every nonempty subset of \( S \) has a minimum element according to \( < \). A sequence is a recurrence if all but finitely many of its terms are defined in terms of its previous terms. Given two sequences \( a_n \) and \( b_n \), we say that \( a_n \) is big Omega of \( b_n \) to mean \( \exists n_0 \in \mathbb{N}, \exists M \in \mathbb{R}, \forall n \geq n_0, M |a_n| \geq |b_n| \). Given two sequences \( a_n \) and \( b_n \), we say that \( a_n \) is big Theta of \( b_n \) to mean that \( a_n \) is big O of \( b_n \) and also \( a_n \) is big Omega of \( b_n \).

3. Suppose that an algorithm has runtime specified by the recurrence relation \( T_n = 2nT_{n/2} + 3 \). Determine what, if anything, the Master Theorem tells us.

Because \( 2n \) is not a constant, the Master theorem does not apply.

4. Use induction to prove that, for all \( n \in \mathbb{N} \), \( \frac{(2n)!}{n!n!} \geq 2^n \).

Base case: \( n = 1 \). \( \frac{(2\cdot1)!}{1!1!} = 2 \), while \( 2^1 = 2 \). Verified.

Inductive case: Let \( n \in \mathbb{N} \), and assume that \( \frac{(2n)!}{n!n!} \geq 2^n \). Multiply by \( \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \). We get

\[
\frac{(2(n+1))!}{(n+1)!(n+1)!} = \frac{(2n+2)(2n+1)(2n)!}{(n+1)(n+1)n!n!} \geq \frac{(2n+2)(2n+1)}{(n+1)(n+1)} \cdot 2^n = \frac{2(2n+1)}{n+1} \cdot 2n = \frac{2^n}{n+1} \cdot 2n+1 = \frac{(n+1)n2^n}{n+1} \geq 2^{n+1}.
\]

Thus \( \frac{(2(n+1))!}{(n+1)!(n+1)!} \geq 2^{n+1} \).

5. Let \( a_n = n^{1.9} + n^2 \). Prove that \( a_n = O(n^2) \).

Take \( n_0 = 1 \) and \( M = 2 \). For all \( n \geq n_0 \), we have \( n^{0.1} \geq 1 = n^0 \), so \( n^2 \geq n^{1.9} \). Hence \( a_n \leq n^2 + n^2 \), and thus \( |a_n| = a_n \leq 2n^2 = 2|n^2| \).

6. Let \( x \in \mathbb{R} \). Prove that there is at most one \( n \in \mathbb{Z} \) with \( n - \frac{1}{2} \leq x < n + \frac{1}{2} \). Do not use any theorems about floors or ceilings.

Suppose that there are \( m, n \in \mathbb{Z} \) with \( n - \frac{1}{2} \leq x < n + \frac{1}{2} \) and \( m - \frac{1}{2} \leq x < m + \frac{1}{2} \). Hence \( n - \frac{1}{2} \leq x < m + \frac{1}{2} \). Adding \( \frac{1}{2} \) to both sides, we get \( n < m + 1 \). But also \( m - \frac{1}{2} \leq x < n + \frac{1}{2} \). Subtracting \( \frac{1}{2} \) from both sides, we get \( m - 1 < n \). Hence \( m - 1 < n < m + 1 \). By Thm 1.12 in the book, since \( m, n \in \mathbb{Z} \), in fact \( m = n \).
7. Let \( x \in \mathbb{R} \). Prove that there is at least one \( n \in \mathbb{Z} \) with \( n - \frac{1}{2} \leq x < n + \frac{1}{2} \). Do not use any theorems about floors or ceilings.

We use maximum element induction. Define \( S = \{ m \in \mathbb{Z} : m - \frac{1}{2} \leq x \} \), a nonempty set of integers with \( x + \frac{1}{2} \) as an upper bound. Hence \( S \) has some maximum element \( n \). \( n - \frac{1}{2} \leq x \) because \( n \in S \). We have two cases: if \( x < n + \frac{1}{2} \), we are done. If instead \( x \geq n + \frac{1}{2} \), then \( n + 1 \) is an integer, and satisfies \((n + 1) - \frac{1}{2} \leq x\), so \( n + 1 \in S \). But then \( n \) was the maximum element of \( S \), a contradiction. Hence \( n - \frac{1}{2} \leq x < n + \frac{1}{2} \).

8. Solve the recurrence, with initial conditions \( a_0 = 3, a_1 = 4 \), and relation \( a_n = 4a_{n-1} - 4a_{n-2} \) \((n \geq 2)\).

This has characteristic polynomial \( r^2 = 4r - 4 \), which factors as \((r - 2)^2 = 0\). Hence we have a double root, and the general solution is \( a_n = A2^n + Bn2^n \). Applying our initial conditions gives \( 3 = a_0 = A2^0 + B \cdot 0 \cdot 2^0 = A \), and \( 4 = a_1 = A2^1 + B \cdot 1 \cdot 2^1 = 2A + 2B \). The system of equations \( \{3 = A, 4 = 2A + 2B\} \) has solution \( \{A = 3, B = -1\} \), so the specific solution is \( a_n = 3 \cdot 2^n - n \cdot 2^n = (3 - n)2^n \).

9. The Tribonacci numbers are given by initial conditions \( T_0 = 0, T_1 = 1, T_2 = 1 \), and recurrence relation \( T_k = T_{k-1} + T_{k-2} + T_{k-3} \) \((k \geq 3)\). Prove that, for all \( k \in \mathbb{N} \), \( T_k < 2^k \).

We handle the three base cases \( k = 0, 1, 2 \) separately: \( T_0 = 0 < 1 = 2^0 \), \( T_1 = 1 < 2 = 2^1 \), \( T_2 = 1 < 4 = 2^2 \). We now use strong induction. Let \( k \in \mathbb{N} \) with \( k \geq 3 \). Assume that \( T_{k-1} < 2^{k-1}, T_{k-2} < 2^{k-2}, T_{k-3} < 2^{k-3} \). Now, since \( k \geq 3 \), \( T_k = T_{k-1} + T_{k-2} + T_{k-3} < 2^{k-1} + 2^{k-2} + 2^{k-3} < 2^{k-1} + 2^{k-2} + 2^{k-3} + 2^{k-3} = 2^{k-1} + 2^{k-2} + 2^{k-2} = 2^{k-1} + 2^{k-1} = 2^k \).

Hence \( T_k < 2^k \).

10. Prove that \( \sqrt{3} \) is irrational.

We argue by contradiction. Suppose that \( \sqrt{3} \) is rational. Hence we may assume there are \( m, n \in \mathbb{Z} \), with \( n \neq 0 \), and \( \sqrt{3} = \frac{m}{n} \). By cancelling any common factors, we may also assume that \( m, n \) have no common factors. Squaring, we get \( 3n^2 = m^2 \) and hence \( 3n^2 = m^2 \). Now, \( 3|m^2 \), and \( 3 \) is prime, so \( 3|m \) (or \( 3|m \)). Write \( m = 3k \), for some integer \( k \), and substitute back. We get \( 3n^2 = (3k)^2 = 9k^2 \). Hence \( n^2 = 3k^2 \). Again, \( 3|n^2 \), and \( 3 \) is prime, so \( 3|n \) (or \( 3|n \)). Hence \( m, n \) both have the common factor \( 3 \), a contradiction.