1. Carefully define the following terms: predicate, $\forall x \in D, P(x)$, counterexample, Proof by Contradiction Theorem.

A predicate is a collection of propositions, indexed by one or more free variables, each drawn from its domain. The expression $\forall x \in D, P(x)$ is a proposition that is $T$ if $P(x)$ is true for every $x \in D$, and $F$ otherwise. A counterexample is an element of a domain that makes a predicate false. The Proof by Contradiction theorem states that for propositions $p, q$, if $(p \land \neg q) \equiv F$, then $p \rightarrow q$ is $T$.

2. Carefully define the following terms: Nonconstructive Existence theorem, Proof by Induction, Proof by Reindexed Induction, Proof by Strong Induction.

The Nonconstructive Existence theorem states that if $(\forall x \in D, \neg P(x)) \equiv F$, then $\exists x \in D, P(x)$ is true. To prove $\forall x \in \mathbb{N}, P(x)$ by induction, we prove both that $P(1)$ is true, and that $\forall x \in \mathbb{N}, P(x) \rightarrow P(x+1)$ is true. To prove $\forall x \in \mathbb{N}, P(x)$ by reindexed induction, we prove both that $P(1)$ is true, and that $\forall x \in \mathbb{N}$ with $x \geq 2$, $P(x-1) \rightarrow P(x)$.

3. Prove that for all $n \in \mathbb{N}$, $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.

Proof by (ordinary) induction on $n$.

Base case ($n = 1$): The LHS has just one summand, namely 1. The RHS is $\frac{1(2)}{2} = 1$.

Inductive case: Assume that $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$. The next summand is $n + 1$, which we add to both sides, to get $\sum_{i=1}^{n+1} i = (n+1) + \sum_{i=1}^{n} i = (n+1) + \frac{n(n+1)}{2} = (n+1)(1+\frac{2}{2}) = (n+1)\frac{2+n}{2} = \frac{(n+1)(n+2)}{2}$.

4. Prove or disprove: $\forall x \in \mathbb{Z}$, $|7x + 20| > 1$.

The statement is false. A counterexample is $x^* = -3$, for which $|7x^* + 20| = |-21 + 20| = |-1| = 1$, which is not strictly greater than 1. In fact, this happens to be the only counterexample.

5. Prove or disprove: $\forall x \in \mathbb{R}$, $\exists y \in \mathbb{R}$, $x^2 < y^2 < x^2 + 1$.

The statement is true. Let $x \in \mathbb{R}$ be arbitrary. We must choose $y$, based on a side calculation. One possible choice is $y = \sqrt{x^2 + \frac{1}{2}}$. Now $y^2 = x^2 + \frac{1}{2}$, and since $x^2 < x^2 + \frac{1}{2} < x^2 + 1$, we get $x^2 < y^2 < x^2 + 1$.

6. Prove or disprove: $\exists y \in \mathbb{R}$, $\forall x \in \mathbb{R}$, $x^2 < y^2 < x^2 + 1$.

The statement is false. To disprove, we let $y \in \mathbb{R}$ be arbitrary. We must now choose $x$, based on a side calculation, to falsify $x^2 < y^2 < x^2 + 1$. One possible choice is $x = y$. This falsifies $x^2 < y^2$, and hence $x^2 < y^2 < x^2 + 1$ (which means $(x^2 < y^2) \land (y^2 < x^2 + 1)$).

7. Let $F_n$ denote the Fibonacci numbers. Prove that $\forall n \in \mathbb{N}$, $F_{2n} = \sum_{i=0}^{n-1} F_{2i+1}$.

This is proved with (ordinary) induction on $n$. 

Base case \((n = 1)\): The LHS is \(F_2 = 1\), while the RHS is a single summand, namely \(F_1 = 1\).

Inductive case: Assume that \(F_{2n} = \sum_{i=0}^{n-1} F_{2i+1}\). The last summand is \(F_{2(n-1)+1} = F_{2n-1}\). The next summand will be \(F_{2n+1}\), so we add this term to both sides, to get \(\sum_{i=0}^{n} F_{2i+1} = F_{2n+1} + \sum_{i=0}^{n-1} F_{2i+1} = F_{2n+1} + F_{2n} = F_{2n+2}\), where we used the Fibonacci recurrence to conclude that \(F_{2n+1} + F_{2n} = F_{2n+2}\).

8. Let \(x \in \mathbb{R}\). Prove that \(2\lfloor x \rfloor \leq \lfloor 2x \rfloor \leq 2\lfloor x \rfloor + 1\).

Solution 1: By a theorem (5.18) in the book, \(\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1\). Now set \(y = x\) to get \(\lfloor x \rfloor + \lfloor x \rfloor \leq \lfloor x + x \rfloor \leq \lfloor x \rfloor + \lfloor x \rfloor + 1\); the desired result follows.

Solution 2: Since \(x \geq \lfloor x \rfloor\), also \(2x \geq 2\lfloor x \rfloor\). We apply a theorem (5.16) in the book to conclude that \(\lfloor 2x \rfloor \geq \lfloor 2\lfloor x \rfloor \rfloor = 2\lfloor x \rfloor\), since \(2\lfloor x \rfloor \in \mathbb{Z}\). Similarly, since \(x \leq \lfloor x \rfloor + 1\), also \(x+x \leq x+\lfloor x \rfloor + 1\), so we again apply theorem 5.16 to conclude that \(\lfloor x + x \rfloor \leq \lfloor x + \lfloor x \rfloor + 1 \rfloor = \lfloor x \rfloor + \lfloor x \rfloor + 1\), by another theorem (5.17).

9. Let \(n \in \mathbb{N}\). Prove that there is at most one \(a \in \mathbb{N}\) satisfying \(a^2 \leq n < (a + 1)^2\).

Suppose that \(a, b \in \mathbb{N}\) with \(a^2 \leq n < (a + 1)^2\) and also \(b^2 \leq n < (b + 1)^2\). We have \(a^2 \leq n < (b + 1)^2\); taking square roots, we conclude that \(a < b + 1\). Similarly, we have \(b^2 \leq n < (a + 1)^2\); taking square roots, we conclude that \(b < a + 1\) and hence \(b - 1 < a\). Combining, we get \(b - 1 < a < b + 1\). Applying a theorem from the book (1.12), since \(a, b\) are integers, we conclude that \(a = b\).

10. Prove that \(\sqrt{5}\) is irrational.

We argue by way of contradiction. We suppose that \(\sqrt{5}\) is rational. We can then express \(\sqrt{5} = \frac{a}{b}\) where \(a, b\) are both integers, \(b \neq 0\), and \(a, b\) have no factors in common. Squaring both sides, we get \(5 = \frac{a^2}{b^2}\) and hence \(5b^2 = a^2\). Thus \(5|a^2\). Since 5 is prime, in fact \(5|a\). Hence there is some integer \(c\) with \(a = 5c\). We substitute to get \(5b^2 = a^2 = (5c)^2 = 25c^2\). Dividing by 5 we get \(b^2 = 5c^2\). Hence \(5|b^2\), and since 5 is prime in fact \(5|b\). But now \(a, b\) have 5 in common as a factor, which is a contradiction.